Hyperbolic Systems: ENO-LLF

I.

Perform one step of first order ENO LLF at the grid point x_j (i.e. going from u_j^n to u_j^{n+1}) for the shallow water equations.

Solution

The shallow water equations are

$$\begin{pmatrix} h \\ hu \end{pmatrix}_t + \begin{pmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \end{pmatrix}_x = \mathbf{0}.$$

Recall from class that the associated Jacobian eigenvalues are

$$\lambda_1 = u + \sqrt{gh}, \lambda_2 = u - \sqrt{gh}$$

with left eigenvectors

$$\mathbf{L}_1 = \begin{pmatrix} \frac{\sqrt{gh}-u}{2\sqrt{gh}}, \frac{1}{2\sqrt{gh}} \end{pmatrix}, \mathbf{L}_2 = \begin{pmatrix} \frac{\sqrt{gh}+u}{2\sqrt{gh}}, -\frac{1}{2\sqrt{gh}} \end{pmatrix}$$

and right eigenvectors

$$\mathbf{R}^1 = \begin{pmatrix} 1 \\ u + \sqrt{gh} \end{pmatrix}, \mathbf{R}^2 = \begin{pmatrix} 1 \\ u - \sqrt{gh} \end{pmatrix}$$

Now, assume we want to update both h and u at x_j. We define $s1$ and $f1$ as

$$s1 = \mathbf{L}_1(u^n_{j+\frac{1}{2}}, h^n_{j+\frac{1}{2}}) \cdot \begin{pmatrix} u \\ hu \end{pmatrix}, f1 = \mathbf{L}_1(u^n_{j+\frac{1}{2}}, h^n_{j+\frac{1}{2}}) \cdot \begin{pmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \end{pmatrix}$$

and similarly $s2$ and $f2$ as

$$s2 = \mathbf{L}_2(u^n_{j+\frac{1}{2}}, h^n_{j+\frac{1}{2}}) \cdot \begin{pmatrix} u \\ hu \end{pmatrix}, f2 = \mathbf{L}_2(u^n_{j+\frac{1}{2}}, h^n_{j+\frac{1}{2}}) \cdot \begin{pmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \end{pmatrix}$$

Then, to perform ENO-LLF we need to evaluate

$$D1^1_i H^+ = \frac{1}{2} f1_i + \frac{1}{2} \alpha 1_{j+\frac{1}{2}} s1_i$$

$$D1^1_i H^- = \frac{1}{2} f1_i - \frac{1}{2} \alpha 1_{j+\frac{1}{2}} s1_i$$

where

$$\alpha 1_{j+\frac{1}{2}} = max(\left| u^n_j + \sqrt{gh^n_j} \right|, \left| u^n_{j+1} + \sqrt{gh^n_{j+1}} \right|).$$

And similarly,

$$D2^1_i H^+ = \frac{1}{2} f2_i + \frac{1}{2} \alpha 2_{j+\frac{1}{2}} s2_i$$

$$D2^1_i H^- = \frac{1}{2} f2_i - \frac{1}{2} \alpha 2_{j+\frac{1}{2}} s2_i$$

where

$$\alpha 2_{j+\frac{1}{2}} = max(\left| u^n_j - \sqrt{gh^n_j} \right|, \left| u^n_{j+1} - \sqrt{gh^n_{j+1}} \right|).$$

We then set

$$F1^+_{j+\frac{1}{2}} = D1^1_j H^+, F1^-_{j+\frac{1}{2}} = D1^1_{j+1} H^-$$

and

$$F2^{+}_{j+\frac{1}{2}} = D2^{1}_{j}H^{+}, F2^{-}_{j+\frac{1}{2}} = D2^{1}_{j+1}H^{-}$$

and finally

$$F1_{j+\frac{1}{2}} = F1^{+}_{j+\frac{1}{2}} + F1^{-}_{j+\frac{1}{2}}$$

$$F2_{j+\frac{1}{2}} = F2^{+}_{j+\frac{1}{2}} + F2^{-}_{j+\frac{1}{2}}.$$

These fluxes are then used as

$$\mathbf{F}_{j+\frac{1}{2}} = F1_{j+\frac{1}{2}}\mathbf{R}_1(u^n_{j+\frac{1}{2}}, h^n_{j+\frac{1}{2}}) + F2_{j+\frac{1}{2}}\mathbf{R}_2(u^n_{j+\frac{1}{2}}, h^n_{j+\frac{1}{2}}).$$

$\mathbf{F}_{j-\frac{1}{2}}$ is computed similarly (e.g. just plug in $j-1$ in place of j in the previous calculation). $\mathbf{F}_{j+\frac{1}{2}}$ and $\mathbf{F}_{j-\frac{1}{2}}$ can then be used to update u^n_j and h^n_j as

$$\begin{pmatrix} u^{n+1}_j \\ h^{n+1}_j u^{n+1}_j \end{pmatrix} = \begin{pmatrix} u^n_j \\ h^n_j u^n_j \end{pmatrix} - \frac{\Delta t}{\Delta x}(\mathbf{F}_{j+\frac{1}{2}} - \mathbf{F}_{j-\frac{1}{2}})$$