Lecture 1

Wednesday, March 30, 2005

1 Introduction

This course is concerned with the numerical solution of partial differential equations. The focus is on hyperbolic PDE. We will also discuss basic methods for numerically solving parabolic and elliptic equations as they relate to hyperbolic problems. Our approach follows the historical development of the numerical methods. Specifically, we will study the numerical solution of equations of types

1. hyperbolic
2. parabolic
3. elliptic

through looking at, respectively, the problems of

A. compressible flow

B. heat flow

C. incompressible flow

2 Convection

Reference: Osher and Fedkiw, Section 3.1

There are two different approaches we can take in describing fluid motion.

1. In the *Lagrangian* formulation, we follow the motion of individual particles as they are advected with the flow. Assuming that a velocity

$\vec{V}(\vec{x}) = \langle u, v, w \rangle$ is known for every point $\vec{x}$ we can compute the particle motion by solving the ordinary differential equation (*ODE*)

$$\frac{d\vec{x}}{dt} = \vec{V}(\vec{x}) \tag{1}$$

This approach is simple, and the equation is easy to solve numerically, but it is hard to apply it to nonlinear phenomena such as shock waves.

2. In the *Eulerian* formulation we use the simple convection (or advection) equation

$$\phi_t + \vec{V} \cdot \nabla\phi = 0 \tag{2}$$

where the t subscript denotes a temporal partial derivative in the time variable t. Recall that ∇ is the gradient operator so that $\vec{V} \cdot \nabla\phi = u\phi_x + v\phi_y + w\phi_z$. This approach can be regarded as sitting still at a point $\vec{x}$ and observing the changes in various quantities at that point due to the flow.

3 Upwind Differencing

Reference: Osher and Fedkiw, Section 3.2

3.1 Notation

In the notes that follow we will always indicate the time step in the superscript, and the spatial indices in the subscripts.

At some point in time, say time t^n, let $\phi^n = \phi(t^n)$ represent the current values of ϕ. Updating ϕ in time consists of finding new values of ϕ at every grid point after some time increment $\triangle t$. We denote these new values of ϕ by $\phi^{n+1} = \phi(t^{n+1})$ where $t^{n+1} = t^n + \triangle t$.

3.2 Temporal Discretization

We look at some possible discretizations for the time derivatives in the above convection equation. The equations below are called *semi-discrete* because we have discretized only the time derivatives.

- *forward Euler*

$$\frac{\phi^{n+1} - \phi^n}{\triangle t} + \vec{V}^n \cdot \nabla\phi^n = 0 \tag{3}$$

- *backward Euler*

$$\frac{\phi^{n+1}-\phi^n}{\triangle t}+\vec{V}^{n+1}\cdot\nabla\phi^{n+1}=0 \tag{4}$$

The forward Euler and backward Euler are both first order accurate, meaning that the error in the discretization is $O(\triangle t)$.

3.3 Spatial Discretization

We begin by writing equation 3 in expanded form as

$$\frac{\phi^{n+1}-\phi^n}{\triangle t}+u^n\phi_x^n+v^n\phi_y^n+w^n\phi_z^n=0 \tag{5}$$

and address the evaluation of the $u^n\phi_x^n$ term first. The techniques used to approximate this term can then be applied independently to the $v^n\phi_y^n$ and $w^n\phi_z^n$ terms in a *dimension by dimension* manner.

For simplicity, consider the one dimensional version of equation 5

$$\frac{\phi^{n+1}-\phi^n}{\triangle t}+u^n\phi_x^n=0 \tag{6}$$

where the sign of u^n indicates whether the values of ϕ are moving to the right or to the left. Since u^n can be spatially varying, we focus on a specific grid point x_i where we write

$$\frac{\phi_i^{n+1}-\phi_i^n}{\triangle t}+u_i^n(\phi_x)_i^n=0 \tag{7}$$

We use the forward Euler discretization in time. This means that we can solve the equations for each grid point independently, and we will not have to solve a linear system at each time step as in the case of backward Euler.

We first introduce the difference operators D^o, D^+ and D^-.

- D^o, *central difference* operator (second-order accurate)

$$(D^o\phi)_i=\frac{\phi_{i+1}-\phi_{i-1}}{2\triangle x} \tag{8}$$

- D^+, *forward difference* operator (first-order accurate)

$$(D^+\phi)_i=\frac{\phi_{i+1}-\phi_i}{\triangle x} \tag{9}$$

- D^-, *backward difference* operator (first-order accurate)

$$(D^- \phi)_i = \frac{\phi_i - \phi_{i-1}}{\triangle x} \tag{10}$$

For the hyperbolic equations, the data is propagated in specific directions (along the characteristic curves). Therefore central differencing, which uses information from both directions, is not very useful for numerically solving hyperbolic problems. Instead we prefer to use *upwinding*. In upwinding the idea is to choose the spatial discretization based on the direction that information is propagating. For the one-dimensional case,

if $u_i > 0$, we use $D^- \phi$
if $u_i < 0$, we use $D^+ \phi$
if $u_i = 0$, then $u_i \phi_x = 0$, so we do not need to approximate ϕ_x

Upwinding is first-order accurate. If we instead choose the difference operators so that data is taken from the direction opposite the one from which it is propagating, then we are using *downwinding* which is unstable.

The numerical errors resulting from upwind differencing cause *dissipation* in the numerical solution. The numerical errors resulting from central differencing cause *dispersion* in the numerical solution (see Strikwerda, Chapter 5).

In any numerical method for solving PDE, we must be concerned with *convergence*. The means that as we refine our grid in time and space, the numerical solution converges to the analytical solution. To this end we examine the notions of *consistency* and *stability*. For precise definitions of convergence, consistency, and stability, see Strikwerda, Chapter 1.

1. Consistency
   The errors in approximating the differential operator vanish as $\triangle t$, $\triangle x \to 0$. This is typically proved by assuming a sufficiently smooth solution and using Taylor series expansion.

2. Stability
   The solution does not blow up in a finite time. Stability may be unconditional or conditional. In conditional stability, we have restrictions on the value of $\triangle t$ so that it be sufficiently small.

For example, it can be shown that discretizing the scalar, linear, constant coefficient convection equation above using central differencing in space and forward Euler in time is consistent, but unstable.

According to the Lax-Richtmyer Equivalence Theorem a finite difference approximation to a linear partial differential equation is convergent if and only if it is both consistent and stable.

A necessary condition for stability is the Courant-Friedreichs-Lewy condition (*CFL* condition) which asserts that the numerical waves should propagate at least as fast as the physical waves. This means that the numerical wave speed of $\triangle x/\triangle t$ must be at least as fast as the physical wave speed $|u|$, i.e. $\triangle x/\triangle t > |u|$. This leads us to the CFL *time step restriction* of

$$\triangle t < \frac{\triangle x}{\max\{|u|\}} \tag{11}$$

where $\max\{|u|\}$ is chosen to be the largest value of $|u|$ over the entire Cartesian grid. Equation 11 is usually enforced by choosing a *CFL number* α with

$$\triangle t \left(\frac{\max\{|u|\}}{\triangle x} \right) = \alpha \tag{12}$$

and $0 < \alpha < 1$. A common near optimal choice is $\alpha = .9$, and a common conservative choice is $\alpha = .5$. A multidimensional CFL condition can be written as

$$\triangle t \max \left\{ \frac{|u|}{\triangle x} + \frac{|v|}{\triangle y} + \frac{|w|}{\triangle z} \right\} = \alpha \tag{13}$$

although

$$\triangle t \left(\frac{\max\{|\vec{V}|\}}{\min\{\triangle x, \triangle y, \triangle z\}} \right) = \alpha \tag{14}$$

is also quite popular.

Instead of upwinding, the spatial derivatives in equation 2 could be approximated with the more accurate central differencing. Unfortunately, simple central differencing is unstable with forward Euler time discretization and the usual CFL conditions with $\triangle t \sim \triangle x$. We look at three possible ways of achieving stability while using central differencing.

1. Stability can be achieved by using a much more restrictive CFL condition with $\triangle t \sim (\triangle x)^2$ although this is too computationally costly.

2. Stability can also be achieved by using a different temporal discretization, e.g. the third order accurate Runge-Kutta method (discussed below).

3. A third way of achieving stability consists of adding some artificial dissipation to the right hand side of equation 2 to obtain

$$\phi_t + \vec{V} \cdot \nabla \phi = \mu \triangle \phi \tag{15}$$

where the viscosity coefficient μ is chosen proportional to $\triangle x$, i.e. $\mu \sim \triangle x$, so that the *artifical viscosity* vanishes as $\triangle x \to 0$ enforcing consistency for this method.

While all three of these approaches stabilize central differencing, we instead prefer to use upwind methods which draw on the highly sucessful technology developed for the numerical solution of conservation laws.