Lecture 10

Monday, May 2, 2005

Supplementary Reading: Osher and Fedkiw, §18.1

1 Incompressible Flow

Recall the stability condition for compressible flow

$$\max_{\Omega}\left\{|u+c|\,,|u|\,,|u-c|\right\} < \frac{\triangle x}{\triangle t}$$

where the quantity on the left of the inequality is the physical wave speed and the quantity on the right is the numerical wave speed. Then the time step is given by

$$\triangle t = \alpha \frac{\triangle x}{\max_{\Omega}\left\{|u+c|\,,|u|\,,|u-c|\right\}}$$

where α is the CFL number, $\alpha < 1$.

For example, we might have $u = 1, c = 300$, so that

$$\begin{aligned} |u+c| &= 301, \\ |u| &= 1, \\ |u-c| &= 299. \end{aligned}$$

Observe that the $u \pm c$ fields impose a much more severe restriction on the time step than the u field. If $|u| \ll |c|$ and we only care about the linear flow phenomena, i.e., the phenomena corresponding to the u field, then we can avoid this difficulty by modeling the flow as incompressible. The assumption of incompressibility is valid in the limit as $\frac{c}{u} \to \infty$ and is equivalent to the divergence free condition $\nabla \cdot \vec{V} = 0$. In fact, the definition of incompressibility for a velocity field $\vec{V}$ is that $\nabla \cdot \vec{V} = 0$.

Modeling the flow as incompressible allows us to eliminate the severe time step restriction due to the $u \pm c$ fields, and focus on the u field. As a result, we lose the nonlinear behavior (e.g., shocks, rarefactions) associated with the $u \pm c$ fields.

2 Equations

Starting from conservation of mass, momentum and energy, the equations for incompressible flow are derived using the divergence free condition, $\nabla \cdot \vec{V} = 0$, which implies that there is no compression or expansion in the flow field.

2.1 Conservation of Mass

In 1D, the equation for conservation of mass is

$$\rho_t + (\rho u)_x = 0$$

Applying the chain rule, we get

$$\rho_t + \rho_x u + \rho u_x = 0$$

Since the flow is incompressible, $\nabla \cdot \vec{V} = 0$ which reduces to $u_x = 0$ in 1D, so that the equation is simply

$$\rho_t + u\rho_x = 0$$

In multiple dimension, the equation is given by

$$\rho_t + \vec{u} \cdot \nabla \rho = 0.$$

2.2 Conservation of Momentum

Starting with the equation for conservation of mass,

$$(\rho u)_t + \left(\rho u^2 + p\right)_x = 0$$

we then apply the chain rule to get

$$\rho_t u + \rho u_t + \rho u u_x + u\,(\rho u)_x + p_x = 0.$$

We combine the first and fourth terms

$$u\,(\rho_t + (\rho u)_x) + \rho u_t + \rho u u_x + p_x = 0.$$

Note that the quantity in parentheses is 0 from conservation of mass, so that

$$\rho u_t + \rho u u_x + p_x = 0.$$

By incompressibility, the second term is 0, so that we are left with

$$\rho u_t + p_x = 0.$$

Dividing by ρ, we get

$$u_t + \frac{p_x}{\rho} = 0. \tag{1}$$

In multiple dimension, the equation is given by

$$\vec{u}_t + \vec{u} \cdot \nabla \vec{u} + \frac{\nabla p}{\rho} = 0.$$

2.3 Conservation of Energy

The equation for conservation of energy in 1D is

$$E_t + [(E + p)\, u]_x = 0.$$

Substituting $E = \rho e + \frac{1}{2}\rho u^2$, we get

$$\left(\rho e + \frac{1}{2}\rho u^2\right)_t + \left[\left(\rho e + \frac{1}{2}\rho u^2 + p\right) u\right]_x = 0.$$

Differentiating, we have

$$\left(e + \frac{1}{2}u^2\right)\rho_t + \rho e_t + \rho u u_t + \left(\rho e + \frac{1}{2}\rho u^2 + p\right) u_x + \left(e + \frac{1}{2}u^2\right) u\rho_x + \rho u e_x + \rho u^2 u_x + u p_x = 0$$

Since $u_x = 0$, this becomes

$$\left(e + \frac{1}{2}u^2\right)\rho_t + \rho e_t + \rho u u_t + \left(e + \frac{1}{2}u^2\right) u\rho_x + \rho u e_x + u p_x = 0$$

Rearranging terms, we have

$$\left(e + \frac{1}{2}u^2\right)(\rho_t + u\rho_x) + u\rho\left(u_t + \frac{p_x}{\rho}\right) + \rho e_t + \rho u e_x = 0$$

By the equations for conservation of mass and conservation of momentum, this reduces to

$$\rho e_t + \rho u e_x = 0$$

Dividing by ρ, we get

$$e_t + u e_x = 0$$

In multiple dimensions the equation for conservation of energy is

$$e_t + \vec{u} \cdot \nabla e = 0$$

In summary, the equations for incompressible flow (in multiple spatial dimensions) are

$$\begin{aligned} \rho_t + \vec{u} \cdot \nabla \rho &= 0 \\ \vec{u}_t + \vec{u} \cdot \nabla \vec{u} + \frac{\nabla p}{\rho} &= 0 \\ e_t + \vec{u} \cdot \nabla e &= 0 \end{aligned}$$

Recall that for compressible flow, we had an equation of state $p = p(\rho, e)$. For incompressible flow, we have $\nabla \cdot \vec{u} = 0$ and do not have an equation of state. Notice also that the equation for conservation of energy is no longer needed to get a closed system. Instead, we have the closed system

$$\begin{aligned} \nabla \cdot \vec{u} &= 0 \\ \rho_t + \vec{u} \cdot \nabla \rho &= 0 \\ \vec{u}_t + \vec{u} \cdot \nabla \vec{u} + \frac{\nabla p}{\rho} &= 0 \end{aligned}$$

In the next lecture, we will see how the pressure is found using an elliptic solver.