Lecture 11

Monday, May 9, 2005

Supplementary Reading: Osher and Fedkiw, §18.2

In the previous lecture we started looking at incompressible flow, where we have the incompressibility assumption, $\nabla \cdot \vec{V} = 0$. Under this assumption, we have a relatively smooth flow, with no shocks or rarefactions (although we may still have contact discontinuities). The equations for conservation of mass, momentum, and energy, assuming incompressibility, are given by

$$\rho_t + \vec{V} \cdot \nabla \rho = 0 \tag{1}$$

$$\vec{V}_t + \vec{V} \cdot \nabla \vec{V} + \frac{\nabla p}{\rho} = 0 \tag{2}$$

$$e_t + \vec{V} \cdot \nabla e = 0 \tag{3}$$

Body forces, e.g. gravity, are added to the RHS of the momentum equation, so that it becomes

$$\vec{V}_t + \vec{V} \cdot \nabla \vec{V} + \frac{\nabla p}{\rho} = \vec{g} \tag{4}$$

where $\vec{g} = (0, g, 0)$.

1 MAC Grid

Harlow and Welch [1] proposed the use of a special grid for incompressible flow computations. This specially defined grid decomposes the computational domain into cells with velocities defined on the cell faces and scalars defined at cell centers. That is, in 2D, $p_{i,j}$, $\rho_{i,j}$ are defined at cell centers while $u_{i\pm\frac{1}{2},j}$ and $v_{i,j\pm\frac{1}{2}}$ are defined at the appropriate cell faces.

Equation (1) is solved by first defining the cell center velocities with simple averaging

$$u_{i,j} = \frac{u_{i-\frac{1}{2},j} + u_{i+\frac{1}{2},j}}{2}$$

$$v_{i,j} = \frac{v_{i,j-\frac{1}{2}} + v_{i,j+\frac{1}{2}}}{2}$$

Then the spatial derivatives are evaluated in a straightforward manner, for example using 3rd order accurate Hamilton-Jacobi ENO. The temporal derivative can be evaluated with a TVD RK scheme.

In order to update the velocity based on equation (2), we first need u and v at all the cell faces. Again, we obtain the values by simple averaging. For example,

$$v_{i-\frac{1}{2},j} = \frac{1}{4}\left(v_{i-1,j-\frac{1}{2}} + v_{i-1,j+\frac{1}{2}} + v_{i,j-\frac{1}{2}} + v_{i,j+\frac{1}{2}}\right).$$

Similarly, to get u values on the v faces, we compute the average

$$u_{i,j-\frac{1}{2}} = \frac{1}{4}\left(u_{i-\frac{1}{2},j-1} + u_{i+\frac{1}{2},j-1} + u_{i-\frac{1}{2},j} + u_{i+\frac{1}{2},j}\right).$$

2 Projection Method

In order to update the velocity, we use the projection method due to Chorin [2]. The projection method is applied by first computing an intermediate velocity field $\vec{V}^\star$ ignoring the pressure term,

$$\frac{\vec{V}^\star - \vec{V}^n}{\triangle t} + \left(\vec{V}^n \cdot \nabla\right)\vec{V}^n = \vec{g}^n, \tag{5}$$

and then computing a divergence free velocity field $\vec{V}^{n+1}$,

$$\frac{\vec{V}^{n+1} - \vec{V}^\star}{\triangle t} + \frac{\nabla p^{n+1}}{\rho^{n+1}} = 0, \tag{6}$$

using the pressure as a correction. Note that combining equations 5 and 6 to eliminate $\vec{V}^\star$ results in equation 4 exactly.

Taking the divergence of equation 6 results in

$$\nabla \cdot \left(\frac{\nabla p^{n+1}}{\rho^{n+1}}\right) = \frac{\nabla \cdot \vec{V}^\star}{\triangle t} \tag{7}$$

after setting $\nabla \cdot \vec{V}^{n+1}$ to zero, i.e. after assuming that the new velocity field is divergence free. Equation 7 defines the pressure in terms of the value of $\triangle t$ used in equation 5. Defining a scaled pressure of $p^\star = p\triangle t$ leads to

$$\vec{V}^{n+1} - \vec{V}^\star + \frac{\nabla p^\star}{\rho^{n+1}} = 0 \tag{8}$$

and

$$\nabla \cdot \left(\frac{\nabla p^\star}{\rho^{n+1}}\right) = \nabla \cdot \vec{V}^\star \tag{9}$$

in place of equations 6 and 7 where $p^\star$ does not depend on $\triangle t$. When the density is spatially constant, we can define $\hat{p} = p\triangle t/\rho$ leading to

$$\vec{V}^{n+1} - \vec{V}^\star + \nabla\hat{p} = 0 \tag{10}$$

and

$$\triangle \hat{p} = \nabla \cdot \vec{V}^\star \tag{11}$$

where $\hat{p}$ does not depend on $\triangle t$ or ρ.

This method utilizes the Helmholtz-Hodge decomposition of the vector field $\vec{V}^\star$,

$$\vec{V}^\star = \vec{V}^{n+1} + \nabla \hat{p}.$$

In general, the Helmholtz-Hodge decomposition of a vector field expresses the vector field as a divergence free vector field plus the gradient of a scalar field.

3 Laplace Equation

In order to discuss the solution of (11), let us first consider solving the Laplace equation.

In 1D, the equation is give by

$$p_{xx} = 0$$

The solution is simply a line

$$p = ax + b \tag{12}$$

The values of the constant a and b are determined by boundary conditions. Assume that the domain is the interval $[0, 1]$. We may have *Dirichlet* boundary conditions, where the value of the function p is given at the boundary. For example,

$$p(0) = p_0$$
$$p(1) = p_1$$

Plugging the boundary conditions in the the equation(12), we get

$$p(0) = b = p_0$$
$$p(1) = a + b = p_1 \Rightarrow a = p_1 - p_0$$

so the coefficients a and b are uniquely determined. Alternatively, *Neumann* boundary conditions specify the value of p_x at the boundary. For example,

$$p_x(0) = 0 \Rightarrow a = 0.$$

This gives us a family of lines with slope 0. To find b, we would need another piece of information. A Dirichlet boundary condition would pick out one of the lines with slope 0, thus determining the solution. But observe that specifying two Neumann conditions could lead to no solution. For example,

$$p_x(0) = 0$$
$$p_x(1) = 1$$

These two boundary conditions are inconsistent, hence there is no solution. Another example is

$$p_x(0) = 0$$
$$p_x(1) = 0$$

In this case, the given boundary conditions are consistent, but incomplete. We still do not have enough information to identify a unique solution. The above examples illustrate the fact that in 1D, for the Laplace equation, we can determine the solution if we have two Dirichlet boundary conditions or one Neumann and one Dirichlet boundary condition, but will have either no solution or an underdetermined solution in the case of two Neumann boundary conditions.

For a discretization in 2D, we can determine the solution if we have either Dirichlet boundary conditions at each point on the boundary, or Neumann boundary conditions at each point except for at least one where we have a Dirichlet boundary condition.

References

[1] Harlow, F. and Welch, J., *Numerical Calculation of Time-Dependent Viscous Incompressible Flow of Fluid with a Free Surface*, The Physics of Fluids 8, 2182-2189 (1965).

[2] Chorin, A., *A Numerical Method for Solving Incompressible Viscous Flow Problems*, J. Comput. Phys. 2, 12-26 (1967).