Lecture 12

Wednesday, May 11, 2005

Supplementary Reading: CS 205 notes on Poisson's equation

1 Laplace Equation

In $1D$ the Laplace equation is given by

$$p_{xx} = 0.$$

The solution is

$$p = ax + b.$$

for some constants a and b. In order to find a and b, we need two boundary conditions. *Dirichlet* boundary conditions specify the value of p at the boundary, e.g.,

$$\begin{cases} p(0) = 0 \\ p(1) = 1 \end{cases} \Rightarrow p(x) = x$$

Neumann boundary conditions specify the derivatives of the function at the boundary. For example, we might have a Neumann boundary condition at $x = 0$ and a Dirichlet boundary condition at $x = 1$,

$$\begin{cases} p_x(0) = 0 \\ p(1) = 1 \end{cases} \Rightarrow p(x) = 1$$

Recall from the previous lecture that if both boundary conditions are of Neumann type, then we either have an inconsistent problem or an underdetermined problem. In the case where we have an underdetermined problem, we still do not know the value of the constant b. However, if only p_x is actually needed in the computation, this may be ok.

2 Numerical Solution

More generally, we are interested in numerically solving Poisson's equation

$$p_{xx} = f(x).$$

At each grid node, we approximate the equation using the second order central difference scheme

$$\frac{p_{i+1} - 2p_i + p_{i-1}}{\triangle x^2} = f_i.$$

The result is a coupled linear system that we need to solve in order to determine p on the entire domain. However, we cannot write this equation as is for the grid points near the boundary since it will involve points outside of the domain. For example, assume that our domain is the interval $[0, 1]$ and that we have grid points $0, 1, \ldots, M, M+1$ uniformly spaced on the domain. The equation for p_1 is

$$\frac{p_2 - 2p_1 + p_0}{\triangle x^2} = f_1.$$

If we have a Dirichlet boundary condition specified on the left of the domain

$$p(0) = \beta,$$

then the equation for p_1 becomes

$$\frac{p_2 - 2p_1}{\triangle x^2} = f_1 - \frac{\beta}{\triangle x^2}.$$

If we have a Neumann boundary condition specified at the half grid point $\frac{1}{2}$

$$p_x(x_{\frac{1}{2}}) = \alpha,$$

we write the equation for p_1 as

$$\frac{\frac{p_2 - p_1}{\triangle x} - \frac{p_1 - p_0}{\triangle x}}{\triangle x} = f_1.$$

Since

$$p_x(x_{\frac{1}{2}}) = \frac{p_1 - p_0}{\triangle x} + O(\triangle x^2),$$

the equation for p_1 becomes

$$\frac{p_2 - p_1}{\triangle x^2} = f_1 + \frac{\alpha}{\triangle x}.$$

Let's look at the matrix equation for the case where we have two Dirichlet boundary conditions.

$$\begin{pmatrix} -2 & 1 & & & & & \\ 1 & -2 & 1 & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & 1 & -2 & 1 & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & 1 & -2 & 1 \\ & & & & & 1 & -2 \end{pmatrix} \begin{pmatrix} p_1 \\ \vdots \\ p_{i-1} \\ p_i \\ p_{i+1} \\ \vdots \\ p_M \end{pmatrix} = \begin{pmatrix} \triangle x^2 f_1 - p_0 \\ \triangle x^2 f_2 \\ \vdots \\ \triangle x^2 f_i \\ \vdots \\ \triangle x^2 f_{M-1} \\ \triangle x^2 f_M - p_{M+1} \end{pmatrix}$$

The matrix is symmetric negative definite. This is advantageous because there are fast linear solvers for such systems, e.g. the conjugate gradients method.

In the case with two Neumann boundary conditions, the matrix equation is

$$\begin{pmatrix} -1 & 1 & & & & & \\ 1 & -2 & 1 & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & 1 & -2 & 1 & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & 1 & -2 & 1 \\ & & & & & 1 & -1 \end{pmatrix} \begin{pmatrix} p_1 \\ \vdots \\ p_{i-1} \\ p_i \\ p_{i+1} \\ \vdots \\ p_M \end{pmatrix} = \begin{pmatrix} \triangle x^2 f_1 - \triangle x \left(p_x\right)_{\frac{1}{2}} \\ \triangle x^2 f_2 \\ \vdots \\ \triangle x^2 f_i \\ \vdots \\ \triangle x^2 f_{M-1} \\ \triangle x^2 f_M - \triangle x \left(p_x\right)_{M+\frac{1}{2}} \end{pmatrix}$$

Notice that the matrix has changed. In particular, it is singular since it has a non-empty null space which is spanned by the vector $(1, \ldots, 1)^T$. This is problematic, but workable. It can be solved for p up to a constant, since for any solution, $\vec{p}$, $\vec{p} + c(1, \ldots, 1)^T$ is also a solution.

In multiple dimension Poisson's equation is

$$\triangle p = f.$$

In $2D$ the equation is

$$p_{xx} + p_{yy} = f.$$

We use the second order accurate central difference discretization

$$\frac{p_{i+1,j} - 2p_{i,j} + p_{i-1,j}}{\triangle x^2} + \frac{p_{i,j+1} - 2p_{i,j} + p_{i,j-1}}{\triangle y^2} = f_{i,j}.$$

In $2D$ we need boundary conditions specified around the entire domain. If at least one boundary condition is Dirichlet, then the resulting matrix will be a banded symmetric positive definite matrix. We can use an iterative solver such as preconditioned conjugate gradients. If all the boundary conditions are Neumann, then the matrix will have a null space.

3 Computing the Pressure

The Laplace and Poisson equations are *elliptic* partial differential equations. Recall that in solving the Navier-Stokes equations using the projection method we derive an elliptic equation for the pressure. To review, the steps in the solution of the momentum equation are

1. Compute the intermediate velocity field $\vec{V}^\star$

$$\frac{\vec{V}^\star - \vec{V}^n}{\triangle t} + \vec{V}^n \cdot \nabla \vec{V}^n = \vec{g} \tag{1}$$

2. Solve an elliptic equation for the pressure

$$\triangle \hat{p} = \nabla \cdot \vec{V}^{\star} \tag{2}$$

3. Compute the divergence free velocity field $\vec{V}^{n+1}$

$$\vec{V}^{n+1} - \vec{V}^{\star} + \nabla \hat{p} = 0 \tag{3}$$

Here we are concerned with step 2, the elliptic solve for the pressure. Specifically, we address the handling of boundary conditions.

Boundary conditions can be applied to either the velocity or the pressure. In order to apply boundary conditions to $\vec{V}^{n+1}$, we apply them to $\vec{V}^{\star}$ after computing $\vec{V}^{\star}$ in equation 1 and before solving equation 2. Then in equation 2, we set $\nabla p \cdot \vec{N} = 0$ on the boundary where $\vec{N}$ is the local unit normal to the boundary. Note that due to the relation in equation 3, this will result in the correct boundary condition for $\vec{V}^{n+1}$.

Recall that the Neumann problem for Poisson's equation must satisfy the *compatibility condition* for a solution to exist. The problem is given by

$$\begin{cases} \triangle p = f & \text{in } \Omega \\ \nabla p \cdot N = g & \text{on } \partial\Omega \end{cases}$$

where $\vec{N}$ is the unit normal to the boundary. From the equation we have the relations

$$\int_{\Omega} f \, dV = \int_{\Omega} \triangle p \, dV = \int_{\Omega} \nabla \cdot \nabla p \, dV = \int_{\partial\Omega} \nabla p \cdot N \, dS = \int_{\partial\Omega} g \, dS$$

where the third equality follows from the divergence theorem. The compatibility condition is

$$\int_{\Omega} f \, dV = \int_{\partial\Omega} g \, dS$$

In solving equation 2, $f = \nabla \cdot \vec{V}^{\star}$ and $g = 0$. Therefore, the compatibility condition is

$$\int_{\Omega} \nabla \cdot \vec{V}^{\star} \, dV = \int_{\partial\Omega} \vec{V}^{\star} \cdot \vec{N} \, dS = 0 \tag{4}$$

where the first equality follows from the divergence theorem. This condition needs to be satisfied when specifying the boundary condition on $\vec{V}^{\star}$ in order to guarantee the existence of a solution.