Lecture 13

Monday, May 16, 2005

Supplementary Reading: Osher and Fedkiw, §18.3, §23.1

1 Projection Method (cont.)

From the previous lectures, we have the following steps in updating the velocity field for incompressible flow using the projection method:

1. Compute the intermediate velocity field $\vec{V}^\star$ (at cell faces)

$$\frac{\vec{V}^\star - \vec{V}^n}{\triangle t} + \vec{V} \cdot \nabla \vec{V} = \vec{g} \tag{1}$$

2. Solve an elliptic equation for the pressure (at cell centers)

$$\nabla \cdot \left(\frac{1}{\rho} \nabla p \right) = \frac{\nabla \cdot \vec{V}^\star}{\triangle t} \tag{2}$$

3. Compute the divergence free velocity field $\vec{V}^{n+1}$ (at cell faces)

$$\frac{\vec{V}^{n+1} - \vec{V}^\star}{\triangle t} + \frac{\nabla p}{\rho} = 0 \tag{3}$$

We can multiply p by $\triangle t$, to get $p^\star = p\triangle t$, and rewrite steps 2 and 3 as

2a.

$$\nabla \cdot \left(\frac{1}{\rho} \nabla p^\star \right) = \nabla \cdot \vec{V}^\star \tag{4}$$

3a.

$$\vec{V}^{n+1} - \vec{V}^\star + \frac{\nabla p^\star}{\rho} = 0 \tag{5}$$

If ρ is constant, then we can move it under the ∇ operator and defining $\hat{p} = \frac{p\triangle t}{\rho}$, we can rewrite steps 2 and 3 as

2b.

$$\triangle \hat{p} = \nabla \cdot \vec{V}^\star \tag{6}$$

3b.

$$\vec{V}^{n+1} - \vec{V}^{\star} + \nabla \hat{p} = 0 \tag{7}$$

Here we discuss the discretization in step 2 of the term $\nabla \cdot \vec{V}^{\star}$. Since we are solving for the pressure, which on the MAC grid lives at the cell centers, we need to discretize the term at the cell centers. We have that

$$\begin{aligned} \left(\nabla \cdot \vec{V}^{\star}\right)_{i,j} &= \left(u_x^{\star} + v_y^{\star}\right)_{i,j} \\ &= \frac{u^{\star}_{i+\frac{1}{2},j} - u^{\star}_{i-\frac{1}{2},j}}{\triangle x} + \frac{v^{\star}_{i,j+\frac{1}{2}} - v^{\star}_{i,j-\frac{1}{2}}}{\triangle y} + O(\triangle x^2) + O(\triangle y^2) \end{aligned}$$

So we have used the intermediate velocity stored at the cell faces to get a second order accurate approximation to $\nabla \cdot \vec{V}^{\star}$ at the cell centers.

Next, we look at computing $\vec{V}^{n+1}$ in step 3b (constant ρ). In $2D$, the equation is

$$\left(u^{n+1}, v^{n+1}\right) - (u^{\star}, v^{\star}) + (\hat{p}_x, \hat{p}_y) = 0$$

where we have written the vectors as row vectors. Therefore we have the two equations

$$\begin{cases} u^{n+1} - u^{\star} + \hat{p}_x = 0 \\ v^{n+1} - v^{\star} + \hat{p}_y = 0 \end{cases}$$

We need a discretization of $(\hat{p}_x, \hat{p}_y)$ at the appropriate cell faces.

$$\begin{aligned} u^{n+1}_{i+\frac{1}{2},j} - u^{\star}_{i+\frac{1}{2},j} + (\hat{p}_x)_{i+\frac{1}{2},j} &= 0 \\ v^{n+1}_{i,j+\frac{1}{2}} - v^{\star}_{i,j+\frac{1}{2}} + (\hat{p}_y)_{i,j+\frac{1}{2}} &= 0 \end{aligned}$$

We have that

$$\begin{aligned} (\hat{p}_x)_{i+\frac{1}{2},j} &= \frac{\hat{p}_{i+1,j} - \hat{p}_{i,j}}{\triangle x} + O(\triangle x^2) \\ (\hat{p}_x)_{i-\frac{1}{2},j} &= \frac{\hat{p}_{i,j} - \hat{p}_{i-1,j}}{\triangle x} + O(\triangle x^2) \end{aligned}$$

Therefore, we have the discretization

$$\begin{aligned} (\triangle \hat{p})_{i,j} = (\nabla \cdot (\nabla \hat{p}))_{i,j} &= \frac{(\hat{p}_x)_{i+\frac{1}{2},j} - (\hat{p}_x)_{i-\frac{1}{2},j}}{\triangle x} + \frac{(\hat{p}_y)_{i,j+\frac{1}{2}} - (\hat{p}_y)_{i,j-\frac{1}{2}}}{\triangle y} + O(\triangle x^2) + O(\triangle y^2) \\ &= \frac{\hat{p}_{i+1,j} - 2\hat{p}_{i,j} + \hat{p}_{i-1,j}}{\triangle x^2} + \frac{\hat{p}_{i,j+1} - 2\hat{p}_{i,j} + \hat{p}_{i,j-1}}{\triangle y^2} + O(\triangle x^2) + O(\triangle y^2). \end{aligned}$$

We have now discussed the details of the discretizations for steps 1, 2, and 3 of the projection method for incompressible flow.

2 Heat Equation

Starting from conservation of mass, momentum and energy one can derive

$$\rho e_t + \rho \vec{V} \cdot \nabla e + p \nabla \cdot \vec{V} = \nabla \cdot (k \nabla T) \tag{8}$$

where

k:	thermal conductivity
T:	temperature
e:	internal energy/unit mass
ρe:	internal energy/unit volume

The assumptions that e and T satisfy the relationship

$$de = c_v dT$$

and that $\nabla \cdot \vec{V} = 0$, simplify equation 8 to

$$\rho c_v T_t + \rho c_v \vec{V} \cdot \nabla T = \nabla \cdot (k \nabla T) \tag{9}$$

which can be further simplified to the standard heat equation

$$\rho c_v T_t = \nabla \cdot (k \nabla T) \tag{10}$$

ignoring the effects of convection, i.e. setting $\vec{V} = 0$. If k is constant, this can be written as

$$T_t = \frac{k}{\rho c_v} \triangle T. \tag{11}$$

Applying explicit Euler time discretization to equation 10 results in

$$\frac{T^{n+1} - T^n}{\triangle t} = \frac{1}{\rho c_v} \nabla \cdot (k \nabla T^n) \tag{12}$$

where either Dirichlet or Neumann boundary conditions can be applied on the boundaries of the computational domain. Assuming that ρ and c_v are constants allows us to rewrite this equation as

$$\frac{T^{n+1} - T^n}{\triangle t} = \nabla \cdot \left(\hat{k} \nabla T^n \right) \tag{13}$$

with $\hat{k} = \frac{k}{\rho c_v}$. Standard central differencing (second order accurate) can be used for the spatial derivatives as in

$$\frac{\hat{k}_{i+\frac{1}{2},j} \left(\frac{T_{i+1,j} - T_{i,j}}{\triangle x} \right) - \hat{k}_{i-\frac{1}{2},j} \left(\frac{T_{i,j} - T_{i-1,j}}{\triangle x} \right)}{\triangle x}$$

A time step restriction of

$$\triangle t \hat{k} \left(\frac{2}{(\triangle x)^2} + \frac{2}{(\triangle y)^2} + \frac{2}{(\triangle z)^2} \right) \leq 1 \tag{14}$$

is needed for stability. If we $\triangle x = \triangle y$, then this is

$$4\frac{\triangle t}{\triangle x^2}\hat{k} \leq 1$$

In $3D$, the restriction is $6\frac{\triangle t}{\triangle x^2}\hat{k} \leq 1$ and in general for nD the restriction is $2n\frac{\triangle t}{\triangle x^2}\hat{k} \leq 1$.

Implicit Euler time discretization

$$\frac{T^{n+1} - T^n}{\triangle t} = \nabla \cdot \left(\hat{k}\nabla T^{n+1}\right) \tag{15}$$

avoids this time step stability restriction. This equation can be rewritten as

$$T^{n+1} - \triangle t \nabla \cdot \left(\hat{k}\nabla T^{n+1}\right) = T^n \tag{16}$$

discretizing the $\nabla \cdot \left(\hat{k}\nabla T^{n+1}\right)$ term using central differencing. For each unknown, T_i^{n+1}, equation 16 is used to fill in one row of a matrix creating a linear system of equations. Since the resulting matrix is symmetric, a number of fast linear solvers can be used (e.g. a PCG method with an incomplete Choleski preconditioner, see Golub and Van Loan [1]). Equation 15 is first order accurate in time and second order accurate in space, and $\triangle t$ needs to be chosen proportional to $\triangle x^2$ in order to obtain an overall asymptotic accuracy of $O(\triangle x^2)$. However, the stability of the implicit Euler method allows one to chose $\triangle t$ proportional to $\triangle x$ saving dramatically on CPU time. The Crank-Nicolson scheme

$$\frac{T^{n+1} - T^n}{\triangle t} = \frac{1}{2}\nabla \cdot \left(\hat{k}\nabla T^{n+1}\right) + \frac{1}{2}\nabla \cdot \left(\hat{k}\nabla T^{n}\right) \tag{17}$$

can be used to achieve second order accuracy in both space and time with $\triangle t$ proportional to $\triangle x$. For the Crank-Nicolson scheme,

$$T^{n+1} - \frac{\triangle t}{2}\nabla \cdot \left(\hat{k}\nabla T^{n+1}\right) = T^n + \frac{\triangle t}{2}\nabla \cdot \left(\hat{k}\nabla T^{n}\right) \tag{18}$$

is used to create a symmetric linear system of equations for the unknowns T_i^{n+1}. Again, all spatial derivatives are computed using standard central differencing.

Why not always use Crank-Nicholson, as it gives second order accuracy and no time step restriction? Let us look at the solution as $\triangle t \to \infty$. Backward Euler gives

$$\triangle T^n = 0,$$

which is the correct steady state solution. Crank-Nicholson gives

$$\triangle T^{n+1} = -\triangle T^n.$$

In $1D$ this is

$$T_{xx}^{n+1} = -T_{xx}^n$$

This shows that the curvature is changing sign at each time step. So the problem with Crank-Nicholson is that as $\triangle t$ gets very large, you get oscillations, whereas with backward Euler, you get the steady-state solution.

References

[1] Golub, G. and Van Loan, C., *Matrix Computations*, The Johns Hopkins University Press, Baltimore, 1989.