Lecture 14

Wednesday, May 18, 2005

In this lecture we will focus on the discretization of the viscosity term in the Navier-Stokes equations. Typically the inviscid equations are called the Euler equations while the viscous equations are called the Navier-Stokes equations.

1 Viscosity

For incompressible flow with nonzero viscosity we still have the same equation for conservation of mass. It is given by

$$\rho_t + \vec{V} \cdot \nabla \rho = 0.$$

However, the momentum equation (in $2D$) becomes

$$\begin{cases} u_t + \vec{V} \cdot \nabla u + \frac{p_x}{\rho} = \frac{(2\mu u_x)_x + (\mu(u_y+v_x))_y}{\rho} \\ v_t + \vec{V} \cdot \nabla v + \frac{p_y}{\rho} = g + \frac{(\mu(u_y+v_x))_x + (2\mu v_y)_y}{\rho} \end{cases} \tag{1}$$

where we have added the viscosity terms to the RHS of the equation. In vector form, this is can be written as

$$\vec{V}_t + \vec{V} \cdot \nabla \vec{V} + \frac{\nabla p}{\rho} = \vec{g} + \frac{(\nabla \cdot \tau)^T}{\rho}$$

where

$$\tau = \mu \begin{pmatrix} 2u_x & u_y + v_x \\ u_y + v_x & 2v_y \end{pmatrix} = \mu \begin{pmatrix} \nabla u \\ \nabla v \end{pmatrix} + \mu \begin{pmatrix} \nabla u \\ \nabla v \end{pmatrix}^T$$

Now consider the special case where $\mu = \mathit{constant}$ in (1). In that case we can simplify the viscosity term on the RHS as follows.

$$\begin{aligned}
\frac{(2\mu u_x)_x + (\mu(u_y+v_x))_y}{\rho} &= \frac{2\mu u_{xx} + \mu u_{yy} + \mu v_{xy}}{\rho} \\
&= \frac{\mu(u_{yy}+u_{xx})}{\rho} + \frac{\mu(u_{xx}+v_{xy})}{\rho} \\
&= \frac{\mu(u_{yy}+u_{xx})}{\rho} + \frac{\mu(u_x+v_y)_x}{\rho} \\
&= \frac{\mu(u_{yy}+u_{xx})}{\rho} + 0 \\
&= \frac{\mu}{\rho}\triangle u
\end{aligned}$$

$$\begin{aligned}
\frac{(\mu(u_y+v_x))_x + (2\mu v_y)_y}{\rho} &= \frac{\mu u_{yx} + \mu v_{xx} + 2\mu v_{yy}}{\rho} \\
&= \frac{\mu(v_{xx}+v_{yy})}{\rho} + \frac{\mu(v_{yy}+u_{xy})}{\rho} \\
&= \frac{\mu(v_{xx}+v_{yy})}{\rho} + \frac{\mu(v_y+u_x)_y}{\rho} \\
&= \frac{\mu(v_{xx}+v_{yy})}{\rho} + 0 \\
&= \frac{\mu}{\rho}\triangle v
\end{aligned}$$

Therefore for $\mu = constant$, the equations (1) become

$$\left\{\begin{array}{l} u_t + \vec{V}\cdot\nabla u + \frac{p_x}{\rho} = \frac{\mu}{\rho}\triangle u \\ v_t + \vec{V}\cdot\nabla v + \frac{p_y}{\rho} = g + \frac{\mu}{\rho}\triangle v \end{array}\right. \tag{2}$$

1.1 Discretization

In the projection method for incompressible flow the viscosity term is included in the computation of $\vec{V}^\star$, the intermediate velocity field. That is, the steps in the projection method become

1. Compute the intermediate velocity field $\vec{V}^\star$

$$\frac{\vec{V}^\star - \vec{V}^n}{\triangle t} + \vec{V}^n\cdot\nabla\vec{V}^n = \frac{(\nabla\cdot\tau)^T}{\rho} + \vec{g} \tag{3}$$

2. Solve an elliptic equation for the pressure

$$\triangle\hat{p} = \nabla\cdot\vec{V}^\star \tag{4}$$

3. Compute the divergence free velocity field $\vec{V}^{n+1}$

$$\vec{V}^{n+1} - \vec{V}^{\star} + \nabla\hat{p} = 0 \tag{5}$$

where we have again assume that $\rho = constant$, and set $\hat{p} = \frac{p\triangle t}{\rho}$.

Next we will discretize the viscous terms in (2). Since we are using a MAC grid and $\vec{V}^{\star}$ is defined at the cell walls, we need the viscous term discretized at the cell walls. We approximate the Laplacian of u at the grid point $i+\frac{1}{2}, j$ as

$$(\triangle u^n)_{i+\frac{1}{2},j} \approx \frac{u^n_{i-\frac{1}{2},j} - 2u^n_{i+\frac{1}{2},j} + u^n_{i+\frac{3}{2},j}}{\triangle x^2} + \frac{u^n_{i+\frac{1}{2},j-1} - 2u^n_{i+\frac{1}{2},j} + u^n_{i+\frac{1}{2},j+1}}{\triangle y^2}$$

This is a second order central difference approximation. The problem with this approximation is that it requires that $\triangle t \sim \triangle x^2$ for stability. This is a severe restriction on the time step and we would like to avoid it. One solution, due to Kim and Moin, is to treat the viscosity implicitly. So for step 1 in the projection method, we solve the equation

$$\frac{\vec{V}^{\star} - \vec{V}^{n}}{\triangle t} + \vec{V}^n \cdot \nabla\vec{V}^n = \frac{(\nabla\cdot\tau^{\star})^T}{\rho} + \vec{g} \tag{6}$$

The term $\vec{V}^n \cdot \nabla\vec{V}^n$ is still treated the same as before. Then the terms at time step n will be on the RHS, while the $\star$ terms are on the LHS. In the case of constant μ, we get a decoupled linear system of the form

$$\begin{cases} A_1 u = b_1 \\ A_2 v = b_2 \end{cases}$$

Another possibility is to use trapezoidal rule

$$\frac{\vec{V}^{\star} - \vec{V}^{n}}{\triangle t} + \vec{V}^n \cdot \nabla\vec{V}^n = \frac{(\nabla\cdot\tau^{\star})^T + (\nabla\cdot\tau^{n})^T}{2\rho} + \vec{g} \tag{7}$$

One problem in incompressible flow is that the numerical viscosity may be larger than the physical viscosity. We want the numerical viscosity arising from the the discretization of the $\vec{V}\cdot\nabla\vec{V}$ term to be smaller than the physical viscosity $\frac{\nabla\cdot\tau}{\rho}$.

Recall the first order upwind discretization of the advection equation

$$u_t + u_x = 0.$$

The discretization is

$$\begin{aligned}
& u_t + \frac{u_i - u_{i-1}}{\triangle x} = 0. \\
\Rightarrow\quad & u_t + \frac{u_i - \left(u_i - \triangle x\,(u_x)_i + \frac{\triangle x^2}{2}\,(u_{xx})_i + O(\triangle x^3)\right)}{\triangle x} = 0 \\
\Rightarrow\quad & u_t + (u_x)_i - \frac{\triangle x}{2}\,(u_{xx})_i = O(\triangle x^2) \\
\Rightarrow\quad & u_t + (u_x)_i = \frac{\triangle x}{2}\,(u_{xx})_i + O(\triangle x^2)
\end{aligned}$$

So we see that the first order upwind discretization for the advection equation gives a second order scheme for the advection-diffusion equation with diffusion coefficient $\frac{\triangle x}{2}$.

Now suppose you want to solve

$$u_t + u_x = \mu u_{xx}.$$

From the above, we see that using a first order upwind discretization for u_x our modified equation will be

$$u_t + u_x = \left(\mu + \frac{\triangle x}{2}\right) u_{xx}.$$

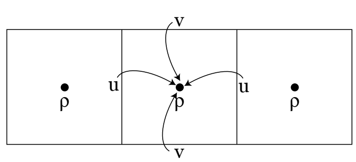
μ is the real viscosity and $\frac{\triangle x}{2}$ is the numerical viscosity.

2 Semi-Lagrangian Advection

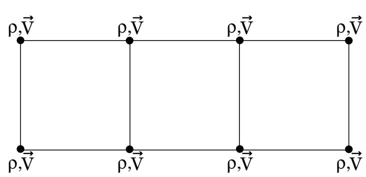
Previously we discretized the equation

$$\rho_t + \vec{V} \cdot \nabla \rho = 0$$

using e.g., 3rd order TVD RK for the temporal derivative and 3rd order ENO in a dimension by dimension approach for the spatial derivative. Here we consider an alternative which is lower order, but not dimension by dimension. It is also called the *method of characteristics*. On the MAC grid, we first average all of the velocities from the faces to the cell centers using standard averaging.

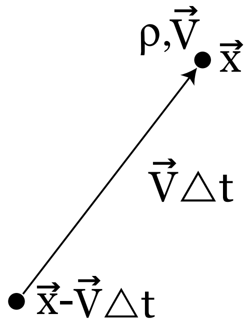


Next, we think of the grid as a regular grid, with all quantities defined at the nodes.



To determine a new value for ρ at time step $n+1$, we look back in the direction $\vec{V}$ a distance $\vec{V}\triangle t$. The new value of ρ is given by

$$\rho^{n+1}(\vec{x}) = \rho^n\left(\vec{x} - \vec{V}\triangle t\right)$$



Generally $\vec{x} - \vec{V} \triangle t$ is not a grid point, so we must use averaging from nearby grid points to get a value for ρ there. Some things to note:

- The method we described is first order. However the method can be made as high order as is desired.
- The method has a very nice stability property. It is unconditionally stable since

$$\max |\rho|^{n+1} \leq \max |\rho|^{n} .$$