Lecture 15

Monday, May 23, 2005

1 Semi-Lagrangian Advection (cont.)

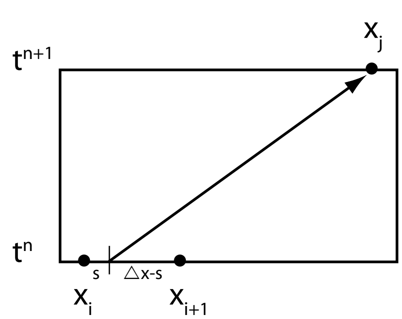
In the previous lecture we described semi-Lagrangian advection (method of characteristics) for solving the advection equation

$$\rho_t + \vec{V} \cdot \nabla \rho = 0$$

where the numerical scheme was given by

$$\rho^{n+1}(\vec{x}) = \rho^n \left(\vec{x} - \vec{V} \triangle t \right)$$

As we noted previously the point $\vec{x} - \vec{V} \triangle t$ is generally not a grid point. To get a value for ρ at that point, we use linear interpolation in $1D$ (bilinear interpolation in $2D$, or trilinear interpolation in $3D$). Let us look at the interpolation in detail in $1D$. Assume that the point $x_j - u \triangle t$ lies between grid points x_i and x_{i+1}, and that its distance from grid point x_i is s.



Therefore its distance from grid point x_{i+1} is $\triangle x - s$. Then linear interpolation gives

$$\begin{aligned} \rho_j^{n+1} &= \rho_i^n + \frac{s}{\triangle x}\left(\rho_{i+1}^n - \rho_i^n\right) \\ &= \frac{\triangle x \rho_i^n + s\rho_{i+1}^n - s\rho_i^n}{\triangle x} \\ &= \frac{\left(\triangle x - s\right)\rho_i^n + s\rho_{i+1}^n}{\triangle x} \end{aligned}$$

or

$$\rho_j^{n+1} = \left(1 - \frac{s}{\triangle x}\right)\rho_i^n + \frac{s}{\triangle x}\rho_{i+1}^n$$

Since $1 - \frac{s}{\triangle x} \leq 1$ and $\frac{s}{\triangle x} \leq 1$, we have that

$$\min\left(\rho_i^n, \rho_{i+1}^n\right) \leq \rho_j^{n+1} \leq \max\left(\rho_i^n, \rho_{i+1}^n\right)$$

This fact is important for unconditional stability. For example, it immediately implies that the method is unconditionally stable in the max norm.

1.1 Semi-Lagrangian Velocity Advection

Recall that for incompressible flow, we have the momentum equation

$$\vec{V}_t + \vec{V} \cdot \nabla \vec{V} + \frac{\nabla p}{\rho} = \vec{g} \tag{1}$$

which in $2D$ is

$$\begin{cases} u_t + \vec{V} \cdot \nabla u + \frac{p_x}{\rho} = 0 \\ v_t + \vec{V} \cdot \nabla v + \frac{p_y}{\rho} = g \end{cases}$$

Using the projection method the steps in the numerical solution are

1. $$\begin{cases} \frac{u^\star - u^n}{\triangle t} + \vec{V}^n \cdot \nabla u^n = 0 \\ \frac{v^\star - v^n}{\triangle t} + \vec{V}^n \cdot \nabla v^n = g \end{cases}$$

2. $$\nabla \cdot \left(\frac{1}{\rho}\nabla p\right) = \frac{\nabla \cdot \vec{V}^\star}{\triangle t}$$

3. $$\begin{cases} \frac{u^{n+1} - u^\star}{\triangle t} + \frac{p_x}{\rho} = 0 \\ \frac{v^{n+1} - v^\star}{\triangle t} + \frac{p_y}{\rho} = 0 \end{cases}$$

Only step 1 has a CFL condition for the hyperbolic terms. The other two steps do not have a time step restriction for stability. Therefore, if we use the semi-Lagrangian method for the velocity advection in step 1 we can eliminate the time step restriction. For $u^\star$, the method is

$$u_j^\star = u^n\left(\vec{x}_j - \vec{V}_j^n \triangle t\right)$$

For $v^\star$ we must also account for gravity, so we have

$$\begin{aligned} v_j^\star &= v^n\left(\vec{x}_j - \vec{V}_j^n \triangle t\right) \\ v_j^\star + &= \triangle t\ g \end{aligned}$$

where we are computing

1) $v_t + \vec{V} \cdot \nabla v = 0$

2) $v_t = g$

This is a Godunov splitting, which is first order accurate.

2 Vorticity

Here we describe a method to counteract the numerical dissipation that damps out many interesting features in the flow.

Taking the curl of the momentum equation (1) gives

$$\vec{\omega}_t + \vec{V} \cdot \nabla \vec{\omega} + \ldots = 0$$

where

$$\vec{\omega} = \nabla \times \vec{V}.$$

In $2D$,

$$\vec{\omega} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & 0 \end{vmatrix} = \left(-\frac{\partial}{\partial z}v, \frac{\partial}{\partial z}u, \frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u\right)$$

Since

$$\frac{\partial}{\partial z}u = \frac{\partial}{\partial z}v = 0$$

we have

$$\begin{aligned} \vec{\omega} &= (0, 0, v_x - u_y) \\ &= (0, 0, \omega) \end{aligned}$$

So this is particularly nice in $2D$ as we get one scalar equation for ω (in $3D$ we still get a 3-vector). Since ω will be either positive or negative, the vorticity vector $\vec{\omega}$ is pointing either into or out of the $x - y$ plane. Vorticity can be thought of as a paddle wheel which is trying to spin the flow. The direction of the spinning depends on the sign of ω.

Some points of interest regarding vorticity are

- Vorticity is <u>conserved</u>.
- Vorticity stays <u>confined</u> in high Reynolds number flows.

Here we discuss a simple turbulence model due to Steinhoff. First we compute vorticity location vectors

$$\vec{N} = \frac{\nabla |\vec{\omega}|}{|\nabla |\vec{\omega}||}.$$

Then we compute the paddle wheel force as

$$\vec{F} = \vec{N} \times \vec{\omega}.$$

Steinhoff's idea was to add a forcing term to the momentum equations

$$\vec{V}_t + \vec{V} \cdot \nabla \vec{V} + \frac{\nabla p}{\rho} = \vec{g} + \epsilon \triangle x \vec{F}$$

It is interesting to note that if you linearize the forcing term, it looks like $-\triangle \vec{V}$.