Lecture 16

Wednesday, May 25, 2005

Consider the linear advection equation

$$\phi_t + \vec{V} \cdot \nabla \phi = 0 \tag{1}$$

or in $1D$

$$\phi_t + u\phi_x = 0.$$

- In <u>upwinding</u>, we choose the discretization of the spatial derivative based on the sign of u.
  - if $u > 0$, we use ϕ_x^- (e.g., $D^-\phi = \frac{\phi_i - \phi_{i-1}}{\triangle x}$)
  - if $u < 0$, we use ϕ_x^+ (e.g., $D^+\phi = \frac{\phi_{i+1} - \phi_i}{\triangle x}$)
  - if $u = 0$, do nothing
- Using <u>ENO</u> we construct higher order approximations to ϕ_x^- and ϕ_x^+.
- Here we describe the <u>WENO</u> (weighted essentially non-oscillatory) method, which gives a better approximation of ϕ_x than ENO.

The following is Osher and Fedkiw, §3.4.

1 Hamilton-Jacobi WENO

When calculating $(\phi_x^-)_i$, the third order accurate HJ ENO scheme uses a subset of $\{\phi_{i-3}, \phi_{i-2}, \phi_{i-1}, \phi_i, \phi_{i+1}, \phi_{i+2}\}$ depending on how the stencil is chosen. In fact, there are exactly three possible HJ ENO approximations to $(\phi_x^-)_i$. Defining $v_1 = D^-\phi_{i-2}$, $v_2 = D^-\phi_{i-1}$, $v_3 = D^-\phi_i$, $v_4 = D^-\phi_{i+1}$ and $v_5 = D^-\phi_{i+2}$ allows us to write

$$\phi_x^1 = \frac{v_1}{3} - \frac{7v_2}{6} + \frac{11v_3}{6} \tag{2}$$

$$\phi_x^2 = -\frac{v_2}{6} + \frac{5v_3}{6} + \frac{v_4}{3} \tag{3}$$

and

$$\phi_x^3 = \frac{v_3}{3} + \frac{5v_4}{6} - \frac{v_5}{6} \tag{4}$$

as the three potential HJ ENO approximations to ϕ_x^-. The goal of HJ ENO is the choose the single approximation with the least error by choosing the smoothest possible polynomial interpolation of ϕ.

In [3], Liu et. al. pointed out that the ENO philosophy of picking exactly one of three candidate stencils is overkill in smooth regions where the data is well behaved. They proposed a Weighted ENO (*WENO*) method that takes a convex combination of the three ENO approximations. Of course, if any of the three approximations interpolate across a discontinuity, it is given minimal weight in the convex combination in order to minimize its contribution and the resulting errors. Otherwise, in smooth regions of the flow, all three approximations are allowed to make a significant contribution in a way that improves the local accuracy from third order to fourth order. Later, Jiang and Shu [2] improved the WENO method by choosing the convex combination weights in order to obtain the optimal fifth order accuracy in smooth regions of the flow. In [1], following the work on HJ ENO in [5], Jiang and Peng extended WENO to the Hamilton-Jacobi framework. This Hamilton-Jacobi WENO or *HJ WENO* scheme turns out to be very useful for solving equation 1 as it reduces the errors by more than an order of magnitude over the third order accurate HJ ENO scheme for typical applications.

The HJ WENO approximation of $(\phi_x^-)_i$ is a convex combination of the approximations in equations 2, 3 and 4 given by

$$\phi_x = \omega_1 \phi_x^1 + \omega_2 \phi_x^2 + \omega_3 \phi_x^3 \tag{5}$$

where the $0 \le \omega_k \le 1$ are the weights with $\omega_1 + \omega_2 + \omega_3 = 1$. The key observation for obtaining high order accuracy in smooth regions is that weights of $\omega_1 = .1$, $\omega_2 = .6$ and $\omega_3 = .3$ give the optimal fifth order accurate approximation to ϕ_x. While this is the optimal approximation, it is only valid in smooth regions. In nonsmooth regions, this optimal weighting can be very inaccurate and we are better off with digital ($\omega_k = 0$ or $\omega_k = 1$) weights that choose a single approximation to ϕ_x, i.e. the HJ ENO approximation.

Reference [2] pointed out that setting $\omega_1 = .1 + O((\triangle x)^2)$, $\omega_2 = .6 + O((\triangle x)^2)$ and $\omega_3 = .3 + O((\triangle x)^2)$ still gives the optimal fifth order accuracy in smooth regions. In order to see this, we rewrite these as $\omega_1 = .1 + C_1(\triangle x)^2$, $\omega_2 = .6 + C_2(\triangle x)^2$ and $\omega_3 = .3 + C_3(\triangle x)^2$ and plug them into equation 5 to obtain

$$.1\phi_x^1 + .6\phi_x^2 + .3\phi_x^3 \tag{6}$$

and

$$C_1(\triangle x)^2 \phi_x^1 + C_2(\triangle x)^2 \phi_x^2 + C_3(\triangle x)^2 \phi_x^3 \tag{7}$$

as the two terms that are added up to give the HJ WENO approximation to ϕ_x. The term given by equation 6 is the optimal approximation which gives the exact value of ϕ_x plus an $O((\triangle x)^5)$ error term. Thus, if the term given by equation 7 is $O((\triangle x)^5)$, then the entire HJ WENO approximation is $O((\triangle x)^5)$ in smooth regions. To see that this is the case, first note that each of the HJ ENO ϕ_x^k approximations gives the exact value of ϕ_x, denoted ϕ_x^E, plus an $O((\triangle x)^3)$ error term (in smooth regions). Thus, the term in equation 7 is

$$C_1(\triangle x)^2\phi_x^E + C_2(\triangle x)^2\phi_x^E + C_3(\triangle x)^2\phi_x^E \tag{8}$$

plus an $O((\triangle x)^2)O((\triangle x)^3) = O((\triangle x)^5)$ term. Since, each of the C_k are $O(1)$, as is ϕ_x^E, this appears to be an $O((\triangle x)^2)$ term at first glance. However, since $\omega_1+\omega_2+\omega_3 = 1$, we have $C_1+C_2+C_3 = 0$ implying that the term in equation 8 is identically zero. Thus, the HJ WENO approximation is $O((\triangle x)^5)$ in smooth regions. Note that [3] obtained only fourth order accuracy, since they chose $\omega_1 = .1 + O(\triangle x)$, $\omega_2 = .6 + O(\triangle x)$ and $\omega_3 = .3 + O(\triangle x)$.

In order to define the weights, ω_k, we follow [1] and estimate the smoothness of the stencils in equations 2, 3 and 4 as

$$S_1 = \frac{13}{12}(v_1 - 2v_2 + v_3)^2 + \frac{1}{4}(v_1 - 4v_2 + 3v_3)^2 \tag{9}$$

$$S_2 = \frac{13}{12}(v_2 - 2v_3 + v_4)^2 + \frac{1}{4}(v_2 - v_4)^2 \tag{10}$$

and

$$S_3 = \frac{13}{12}(v_3 - 2v_4 + v_5)^2 + \frac{1}{4}(3v_3 - 4v_4 + v_5)^2 \tag{11}$$

respectively. Using these smoothness estimates, we define

$$\alpha_1 = \frac{.1}{(S_1+\epsilon)^2} \tag{12}$$

$$\alpha_2 = \frac{.6}{(S_2+\epsilon)^2} \tag{13}$$

and

$$\alpha_3 = \frac{.3}{(S_3+\epsilon)^2} \tag{14}$$

with

$$\epsilon = 10^{-6}\max\left\{v_1^2, v_2^2, v_3^2, v_4^2, v_5^2\right\} + 10^{-99} \tag{15}$$

where the 10^{-99} term is set to avoid division by zero in the definition of the α_k. This value for epsilon was first proposed by Fedkiw et al. [4] where the first term is a scaling term that aids in the balance between the optimal fifth order accurate stencil and the digital HJ ENO weights. In the case that ϕ is an approximate signed distance function, the v_k which approximate ϕ_x are approximately equal to one so that the first term in equation 15 can be set to 10^{-6}. This first term can then absorb the second term yielding $\epsilon = 10^{-6}$ in place of equation 15. Since the first term in equation 15 is only a scaling term, it is valid to make this $v_k \approx 1$ estimate in multidimensions as well.

A smooth solution has small variation leading to small S_k. If the S_k are small enough compared to ϵ then equations 12, 13 and 14 become $\alpha_1 \approx .1\epsilon^{-2}$, $\alpha_2 \approx .6\epsilon^{-2}$ and $\alpha_3 \approx .3\epsilon^{-2}$ exhibiting the proper ratios for the optimal fifth order accuracy. That is, normalizing the α_k to obtain the weights

$$\omega_1 = \frac{\alpha_1}{\alpha_1 + \alpha_2 + \alpha_3} \tag{16}$$

$$\omega_2 = \frac{\alpha_2}{\alpha_1 + \alpha_2 + \alpha_3} \tag{17}$$

and

$$\omega_3 = \frac{\alpha_3}{\alpha_1 + \alpha_2 + \alpha_3} \tag{18}$$

gives (approximately) the optimal weights of $\omega_1 = .1$, $\omega_2 = .6$ and $\omega_3 = .3$ when the S_k are small enough to be dominated by ϵ. Nearly optimal weights are also obtained when the S_k are larger than ϵ as long as all the S_k are approximately the same size as is the case for sufficiently smooth data. On the other hand, if the data is not smooth as indicated by large S_k, then the corresponding α_k will be small compared to the other α_k's giving that particular stencil limited influence. If two of the S_k are relatively large, then their corresponding α_k's will both be small and the scheme will rely most heavily on a single stencil similar to the digital behavior of HJ ENO. In the unfortunate instance that all three of the S_k are large, the data is poorly conditioned and none of the stencils are particularly useful. This case is problematic for the HJ ENO method as well, but fortunately it usually occurs only locally in space and time allowing the methods to repair themselves after the situation subsides.

$(\phi_x^+)_i$ is constructed with a subset of $\{\phi_{i-2}, \phi_{i-1}, \phi_i, \phi_{i+1}, \phi_{i+2}, \phi_{i+3}\}$. Defining $v_1 = D^+\phi_{i+2}$, $v_2 = D^+\phi_{i+1}$, $v_3 = D^+\phi_i$, $v_4 = D^+\phi_{i-1}$ and $v_5 = D^+\phi_{i-2}$ allows us to use equations 2, 3 and 4 as the three HJ ENO approximations to $(\phi_x^+)_i$. Then the HJ WENO convex combination is given by equation 5 with the weights given by equations 16, 17 and 18.

References

[1] Jiang, G.-S. and Peng, D., *Weighted ENO Schemes for Hamilton Jacobi Equations*, SIAM J. Sci. Comput. 21, 2126-2143 (2000).

[2] Jiang, G.-S. and Shu, C.-W., *Efficient Implementation of Weighted ENO Schemes*, J. Comput. Phys. 126, 202-228 (1996).

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[4] Fedkiw, R., Merriman, B., and Osher, S., *Simplified Upwind Discretization of Systems of Hyperbolic Conservation Laws Containing Advection Equations*, J. Comput. Phys. 157, 302-326 (2000).

[5] Osher, S. and Shu, C.-W., *High Order Essentially Non-Oscillatory Schemes for Hamilton-Jacobi Equations*, SIAM J. Numer. Anal. 28, 902-921 (1991).