Lecture 3

Wednesday, April 6, 2005

Supplementary Reading: Osher and Fedkiw, Section 14.1

Previously we motivated our study of numerical methods for hyperbolic conservation laws by focusing on the linear advection equation, $\phi_t + \vec{V} \cdot \nabla \phi = 0$. We looked at various methods, including 1st, 2nd, and 3rd order HJ ENO for the spatial derivative and 1st, 2nd, and 3rd order TVD RK for the temporal derivative. We now shift our focus to solving hyperbolic conservation laws.

1 Hyperbolic Conservation Laws

A continuum physical system is described by the laws of conservation of mass, momentum and energy. The integral form of the conservation law is derived by considered a fixed *control volume* (or region in two dimensions). Let us denote the control volume by Ω, and its boundary by $\partial\Omega$. if u represents the conserved quantity, then the total amount of u in the control volume is given by

$$\int_{\Omega} u \; dV$$

The rate of change of the total amount in the control volume is given by the flux through the region boundary, plus whatever internal sources exist.

$$\frac{d}{dt}\int_{\Omega} u \; dV = -\int_{\partial\Omega} \vec{f}(u) \cdot dA + \int_{\Omega} s(u) \; dV$$

The flux can be either convective or diffusive. The distinction is that diffusive fluxes are driven by gradients, while convective fluxes persist even in the absence of gradients. As an example of a diffusive flux, consider the opening of a perfume bottle. The gradient in concentration of the perfume causes it to diffuse. For most flows where compressibility is important, e.g.

flows with shock waves, one only needs to model the convective transport and can ignore diffusion (mass diffusion, viscosity and thermal conductivity) as well as the source terms (such as chemical reactions, atomic excitations, and ionization processes). Moreover, convective transport requires specialized numerical treatment while diffusive and reactive effects can be treated with standard numerical methods, such as simple central differencing, that are independent of those for the convective terms. A source term might include creation of the quantity through a chemical reaction. Conservation laws with only convective fluxes are known as hyperbolic conservation laws.

The *weak form* of the conservation law is usually written as

$$\frac{d}{dt}\int_{\Omega} u \, dV + \int_{\partial\Omega} \vec{f}(u) \cdot dA = \int_{\Omega} s(u) \, dV$$

The equation now resembles the linear advection equation we looked at previously.

We now consider the *strong form*, or differential form of the conservation law. The strong form can be derived from the weak form by taking an infinitesimally small control volume and applying the divergence theorem. The equation is then written as

$$\frac{\partial u}{\partial t} + \nabla \cdot \vec{f}(u) = s(u)$$

The strong form may not always hold, as it requires that $\frac{\partial u}{\partial t}$ and $\nabla \cdot \vec{f}(u)$ exist. The strong form is not valid when there is a shock, contact discontinuity, or when the function is not smooth. These are the types of phenomena we would like to consider.

The first thing to realize is that the presence of discontinuities poses a limitation on the order of accuracy of any numerical scheme we might devise. There is a conjecture that states that we cannot get a scheme with higher than first order accuracy. However, we are still interested in higher order methods such as ENO, because although our scheme is limited to first order accuracy overall, in many parts of the domain the dominant error term will be the higher order one. For example, we may have an error that looks like $C_1 \triangle x + C_2 \triangle x^3$, with $C_1 \ll C_2$ almost everywhere in our domain. This is called a *high resolution* method.

The important physical phenomena exhibited by hyperbolic conservation laws are

1. bulk convection and waves
2. contact discontinuities

3. shocks

4. rarefactions.

We briefly describe the physical features and mathematical model equations for each effect, and most importantly note the implications they have on the design of numerical methods.

1.1 Bulk Convection and Waves

Bulk convection is simply the bulk movement of matter, carrying it from one spot to another, like water streaming from a hose. Waves are small amplitude smooth disturbances that transmit through the system without any bulk transport like ripples on a water surface or sound waves through air. Whereas convective transport occurs at the gross velocity of the material, waves propagate at the "speed of sound" in the system (relative to the bulk convective motion of the system). Waves interact by superposition, so that they can either cancel out (interfere) or enhance each other.

The simplest model equation that describes bulk convective transport is the linear convection equation

$$u_t + \vec{v} \cdot \nabla u = 0 \tag{1}$$

where $\vec{v}$ is a constant equal to the convection velocity. The solution to this is simply that u translates at the constant speed $\vec{v}$. This same equation can also be taken as a simple model of wave motion, if u is a sine wave and $\vec{v}$ is interpreted as the speed of sound. The linear convection equation is also an important model for understanding smooth transport in any conservation law. As long as $\vec{f}$ is smooth and u has no jumps in it, the general scalar conservation law

$$u_t + \nabla \cdot \vec{f}(u) = 0 \tag{2}$$

can be rewritten as

$$u_t + \vec{f}'(u) \cdot \nabla u = 0 \tag{3}$$

where $\vec{f}'(u)$ acts as a convective velocity. That is, locally in smooth parts of the flow, a conservation law behaves like bulk convection with velocity $\vec{f}'(u)$. This is called the local characteristic velocity of the flow. For systems, the term $\vec{f}'(u)$ is the Jacobian $\frac{\partial \vec{f}}{\partial \vec{u}}$.

Note: One must be careful in going from equation from (2) to (3). In doing so, we are assuming that f depends on x through u only. For example, consider the equation for conservation of mass in one dimension.

$$\rho_t + (\rho u)_x = 0$$

The chain rule gives

$$\rho_t + u\rho_x + \rho u_x = 0$$

However, applying (3) with $f(\rho) = \rho u$ would give

$$\rho_t + u\rho_x = 0$$

which is a linearization. It assumes that $u_x = 0$, or that f depends on x through ρ only. The linearized equation is incorrect. However, it can be used as an aid in developing intuition and as a guide for devising numerical schemes.

1.2 Contact Discontinuities

A contact discontinuity is a persistent, discontinuous jump in mass density moving by bulk convection through the system. Since there is negligible mass diffusion, such a jump persists. These jumps usually appear at the point of contact of different materials, for example, a contact discontinuity separates oil from water. Contacts move at the local bulk convection speed, or more generally the characteristic speed, and can be modeled by using step-function initial data in the bulk convection equation 1. Since contacts are simply a bulk convection effect, they retain any perturbations they receive. Thus we expect contacts to be especially sensitive to numerical methods, i.e. any spurious alteration of the contact will tend to persist and accumulate.

Note that there is no discontinuity in pressure or velocity across the contact discontinuity, but only in density.