

# Lecture 4

Monday, April 11, 2005

**Supplementary Reading:** Osher and Fedkiw, §14.1.3, §14.1.4; Leveque §11.6, §12.9, §12.10, §12.11

## 1 Hyperbolic Conservation Laws

The important physical phenomena exhibited by hyperbolic conservation laws are

1. bulk convection and waves
2. contact discontinuities
3. shocks
4. rarefactions

The first two phenomena are called "linearly degenerate" because they can be modeled locally by the linear advection equation, whereas the last two phenomena are called "genuinely nonlinear".

In the previous lecture we discussed bulk convection and waves and contact discontinuities. In this lecture we look at shocks and rarefactions.

### 1.1 Shock Waves

A shock is a spatial jump in material properties, like pressure and temperature, that develops spontaneously from smooth distributions and then persists. The shock jump is self-forming and also self-maintaining. This is unlike a contact discontinuity which must be put in the system initially and will not re-sharpen itself if it is smeared out by some other process. Shocks develop through a feedback mechanism in which strong impulses move faster than weak ones, and thus tend to steepen themselves up into a "step" profile as they travel through the system. Familiar examples are the "sonic boom" of a jet aircraft, or the "bang" from a gun. These sounds are our perceptions of a sudden jump in air pressure.

The simplest model equation that describes shock formation is the one dimensional Burgers' equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0$$

which looks like the convection equation with a non-constant convective speed of  $u$ , i.e.  $u_t + uu_x = 0$ . Note that this equation is nonlinear, because the velocity depends on the conserved variable  $u$ . Thus larger  $u$  values move faster, and they will overtake smaller values. This ultimately resulting in the development of, for example, a right-going shock if the initial data for  $u$  is any positive, decreasing function. Shocks move at a speed that is not simply related to the bulk flow speed or characteristic speed, and is not immediately evident from examining the flux, in contrast to contacts. Shock speed is controlled by the difference between influx and outflux of conserved quantity into the region. Specifically, suppose a conserved quantity  $u$  with conservation law

$$u_t + f(u)_x = 0$$

has a step function profile with constant values extending both to the left,  $u_L$ , and to the right,  $u_R$ , with a single shock jump transition in between moving with speed  $s$ . Then the integral form of the conservation law,

$$\frac{d}{dt} \int_{\Omega} u \, dV = - \int_{\partial\Omega} \vec{f}(u) \cdot dA$$

applied to any interval containing the shock, gives the relation

$$\begin{aligned} \frac{s\Delta t(u_R - u_L)}{\Delta t} &= f(u_R) - f(u_L) \\ \Rightarrow s(u_R - u_L) &= f(u_R) - f(u_L) \end{aligned}$$

which is just another statement that the rate at which  $u$  appears,  $s(u_R - u_L)$ , in the interval of interest is given by the difference in fluxes across the interval. See figure 1 below.

We use the notation  $[u]$  to denote the jump in  $u$ . The shock speed can then be written

$$s = \frac{[f(u)]}{[u]}$$

for the scalar conservation law, or

$$s[u] = [f(u)]$$

for a system of conservation laws.

For example, we look at Burgers equation with  $u_L = 1$  and  $u_R = -1$ . We

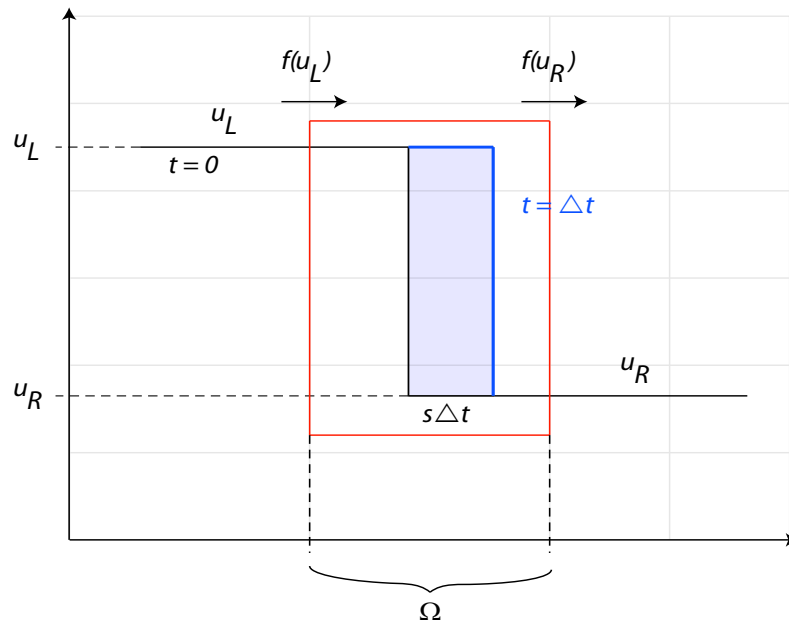


Figure 1: Derivation of the shock speed for a conserved quantity  $u$  with a step function profile with constant values extending both to the left,  $u_L$ , and to the right,  $u_R$ . The red box signifies a control region. The blue shaded region is the additional amount of  $u$  in the control region due to the motion of the shock.

compute the shock speed.

$$\begin{aligned} f(u_L) &= \frac{u_L^2}{2} = \frac{1}{2} \\ f(u_R) &= \frac{u_R^2}{2} = \frac{1}{2} \\ s &= \frac{[f(u)]}{[u]} \\ &= \frac{0}{2} = 0 \end{aligned}$$

The computation shows that the shock sits still. If we take  $u_L = 2$  and  $u_R = -1$ , we have

$$\begin{aligned} f(u_L) &= \frac{u_L^2}{2} = \frac{4}{2} = 2 \\ f(u_R) &= \frac{u_R^2}{2} = \frac{1}{2} \\ s &= \frac{[f(u)]}{[u]} \\ &= \frac{\frac{1}{2} - 2}{-1 - 2} = \frac{-1.5}{-3} = \frac{1}{2} \end{aligned}$$

Thus we see that the proper speed of the shock is directly determined by conservation of  $u$  via the flux  $f$ . This has an important implication for numerical method design: namely, a numerical method will only “capture” the correct shock speeds if it has “conservation form”, i.e. if the rate of change of  $u$  at some node is the difference of fluxes which are accurate approximations of the real flux  $f$ . The *Lax-Wendroff theorem* tells us that if a consistent and conservative method converges, then it converges to a weak solution of the conservation law (see Leveque, §12.10). Note that a weak solution may not be unique. Typically we are interested in the weak solution that satisfies an *entropy condition* (see Leveque, §12.11).

In the presence of shocks and other discontinuities, we must look for weak solutions to the conservation law. An alternative is to modify the differential equation by adding a small amount of viscosity.

$$u_t + f(u)_x = \epsilon u_{xx}, \quad \epsilon > 0$$

As  $\epsilon \rightarrow 0$ , we obtain the *vanishing viscosity* solution. This solution is the unique weak solution that satisfies the entropy condition. See Leveque, §11.6 for more details about the vanishing viscosity solution.

Shocks have a self-sharpening feature that has two implications for numerical methods. First, it means that even if the initial data is smooth, steep gradients and jumps will form spontaneously. Thus, our numerical method must be prepared to deal with shocks even if none are present in the initial data. Second, there is a beneficial effect from self-sharpening, because modest numerical

errors introduced near a shock (smearing or small oscillations) will tend to be eliminated, and will not accumulate. The shock is naturally driven towards its proper shape. Because of this, computing strong shocks is mostly a matter of having a conservative scheme in order to get their speed correct.

## 1.2 Rarefactions

Whereas Burgers equation with monotonically decreasing initial data results in the formation of a shock, Burgers equation with monotonically increasing initial data results in the formation of a *rarefaction*. A rarefaction is a discontinuous jump or steep gradient in properties that dissipates as a smooth expansion. A common example is the jump in air pressure from outside to inside a balloon which dissipates as soon as the balloon is burst and the high pressure gas inside is allowed to expand. Such an expansion also occurs when the piston in an engine is rapidly pulled outward from the cylinder.

A rarefaction tends to smooth out local features which is generally beneficial for numerical modeling. However, a rarefaction often connects to a smooth (e.g. constant) solution region and this results in a “corner” which is notoriously difficult to capture accurately. The main numerical problem posed by rarefactions is that of initiating the expansion. If the initial data is a perfect, symmetrical step, such as  $u(x) = \text{sign}(x)$ , it may be “stuck” in this form, since the steady state Burgers’ equation is satisfied identically (i.e. the flux  $u^2/2$  is constant everywhere, and similarly in any numerical discretization). However, local analysis can identify this stuck expansion, because the characteristic speed  $u$  on either side points away from the jump suggesting its potential to expand. In order to get the initial data unstuck, a small amount of smoothing must be applied to introduce some intermediate state values that have a non-constant flux to drive expansion. In numerical methods, this smoothing applied at a jump where the effective local velocity indicates expansion should occur is called an “entropy fix”, since it allows the system to evolve from the artificial low entropy initial state to the proper increased entropy state of a free expansion.

Let us illustrate the point above. Assume we are solving Burgers equation with initial data  $u_L = -1$  and  $u_R = 1$ . Then

$$\begin{aligned} f(u_R) &= \frac{1}{2} \\ f(u_L) &= \frac{1}{2} \\ \Rightarrow u_t &= 0 \end{aligned}$$

In this case our numerical solution is “stuck”, and we are computing an entropy violating expansion shock. We don’t want this solution. In order to obtain the correct, entropy condition satisfying solution, which is the rarefaction, we add some numerical smearing by solving

$$u_t + \left(\frac{u^2}{2}\right)_x = \epsilon u_{xx}$$

so that initially we are solving

$$u_t = \epsilon u_{xx}$$

This smears out the step function profile enough to initiate the rarefaction.

Note: One challenge in computing numerical solutions to hyperbolic conservation laws is that a wrong solution might look very good visually, but still be incorrect. For example, if your scheme is not in conservation form, the solution might look almost correct, except that the location of the shock will be off by a few grid cells. This sort of error can be difficult or impossible to detect visually. In short, make sure your scheme is in conservation form!