Lecture 6

Monday, April 18, 2005

Supplementary Reading: Osher and Fedkiw, §14.3.2, §14.3.3

In the previous lecture we introduced the numerical flux function. To review, we start with the strong form of the conservation law,

$$u_t + f(u)_x = 0.$$

Integrating over a grid cell, we have the weak form

$$(u_{ave,i}\triangle x)_t + f\left(u_{i+\frac{1}{2}}\right) - f\left(u_{i-\frac{1}{2}}\right) = 0.$$

Replacing $u_{ave,i}$ with the pointwise value u_i we make an $O(\triangle x^2)$ error

$$(u_i\triangle x)_t + f\left(u_{i+\frac{1}{2}}\right) - f\left(u_{i-\frac{1}{2}}\right) = O(\triangle x^2)$$

Introducing the numerical flux function instead of the physical flux function eliminates the error

$$(u_i)_t + \frac{\mathcal{F}\left(x_{i+\frac{1}{2}}\right) - \mathcal{F}\left(x_{i-\frac{1}{2}}\right)}{\triangle x} = 0.$$

1 Constructing the Numerical Flux Function

We define the numerical flux function through the relation

$$f(u_i)_x = \frac{\mathcal{F}_{i+1/2} - \mathcal{F}_{i-1/2}}{\Delta x} \tag{1}$$

To obtain a convenient algorithm for computing this numerical flux function, we define $h(x)$ implicitly through the following equation

$$f(u(x)) = \frac{1}{\triangle x}\int_{x-\triangle x/2}^{x+\triangle x/2} h(y)dy \tag{2}$$

and note that taking a derivative on both sides of this equation yields

$$f(u(x))_x = \frac{h(x + \triangle x/2) - h(x - \triangle x/2)}{\triangle x} \tag{3}$$

which shows that h is identical to the numerical flux function at the cell walls. That is $\mathcal{F}_{i\pm 1/2} = h(x_{i\pm 1/2})$ for all i. We calculate h by finding its primitive

$$H(x) = \int_{x_{-1/2}}^{x} h(y)dy \tag{4}$$

using polynomial interpolation, and then take a derivative to get h. We build a divided difference table to construct H.

zeroth order	$D^0_{i+\frac{1}{2}}H$	at cell walls
first order	$D^1_i H$	at cell centers
second order	$D^2_{i+\frac{1}{2}}H$	at cell walls
third order	$D^3_i H$	at cell centers
$\vdots$	$\vdots$	$\vdots$

That is, the even divided differences of H are at the cell walls, and the odd divided differences of H are at the cell centers. Since we are actually interested in determining h, we do not need the zeroth order divided differences of H as they will drop out when we differentiate to obtain h. Therefore, we can ignore the zeroth level of the divided difference table for H, and construct the table starting at the first level. The first level is given by

$$\begin{aligned} D^1_i H &= \frac{H\left(x_{i+\frac{1}{2}}\right) - H\left(x_{i-\frac{1}{2}}\right)}{\triangle x} \\ &= f\left(u_i\right) \\ &= D^0_i f \end{aligned}$$

This is because

$$\begin{aligned} H(x_{i+\frac{1}{2}}) &= \int_{x_{-1/2}}^{x_{i+1/2}} h(y)dy \\ &= \sum_{j=0}^{i} \left(\int_{x_{j-1/2}}^{x_{j+1/2}} h(y)dy \right) \\ &= \triangle x \sum_{j=0}^{i} f(u(x_j)) \end{aligned}$$

And similarly,

$$H(x_{i-\frac{1}{2}}) = \triangle x \sum_{j=0}^{i-1} f(u(x_j))$$

So that

$$H(x_{i+\frac{1}{2}}) - H(x_{i-\frac{1}{2}}) = \triangle x f(u(x_i))$$

The higher divided differences are

$$D^2_{i+1/2}H = \frac{f(u(x_{i+1})) - f(u(x_i))}{2\triangle x} = \frac{1}{2}D^1_{i+1/2}f \tag{5}$$

$$D^3_i H = \frac{1}{3}D^2_i f \tag{6}$$

continuing in that manner.

According to the rules of polynomial interpolation, we can take any path along the divided difference table to construct H, although they do not all give good results. ENO reconstruction consists of two important features. First, choose $D^1_i H$ in the upwind direction. Second, choose higher order divided differences by taking the smaller in absolute value of the two possible choices. Once we construct $H(x)$, we evaluate $H'(x_{i+1/2})$ to get the numerical flux $\mathcal{F}_{i+1/2}$.

2 ENO-Roe Discretization (Third Order Accurate)

For a specific cell wall, located at $x_{i_0+1/2}$, we find the associated numerical flux function $\mathcal{F}_{i_0+1/2}$ as follows. First, we define a *characteristic speed*

$$\lambda_{i_0+1/2} = f'(u_{i_0+1/2})$$

For example, recall Burgers' equation,

$$u_t + \left(\frac{u^2}{2}\right)_x = 0.$$

The flux is given by

$$f(u) = \left(\frac{u^2}{2}\right)$$

so that

$$f'(u) = u$$

Therefore,

$$\lambda(x) = f'(u(x)) = u(x).$$

The value of u at the half grid points is defined using a standard linear average

$$u_{i_0+1/2} = (u_{i_0} + u_{i_0+1})/2$$

Then, if $\lambda_{i_0+1/2} > 0$, set $k = i_0$. Otherwise, set $k = i_0 + 1$. Define

$$Q_1(x) = (D_k^1 H)(x - x_{i_0+1/2}) \tag{7}$$

If $|D^2_{k-1/2}H| \leq |D^2_{k+1/2}H|$, then $c = D^2_{k-1/2}H$ and $k^\star = k - 1$. Otherwise, $c = D^2_{k+1/2}H$ and $k^\star = k$. Define

$$Q_2(x) = c(x - x_{k-1/2})(x - x_{k+1/2}) \tag{8}$$

If $|D^3_{k^\star}H| \leq |D^3_{k^\star+1}H|$, then $c^\star = D^3_{k^\star}H$. Otherwise, $c^\star = D^3_{k^\star+1}H$. Define

$$Q_3(x) = c^\star(x - x_{k^\star-1/2})(x - x_{k^\star+1/2})(x - x_{k^\star+3/2}) \tag{9}$$

Then

$$\mathcal{F}_{i_0+1/2} = H'(x_{i_0+1/2}) = Q_1'(x_{i_0+1/2}) + Q_2'(x_{i_0+1/2}) + Q_3'(x_{i_0+1/2}) \tag{10}$$

which simplifies to

$$\mathcal{F}_{i_0+1/2} = D_k^1 H + c\,(2(i_0 - k) + 1)\,\triangle x + c^\star \left(3(i_0 - k^\star)^2 - 1\right)(\triangle x)^2. \tag{11}$$