Lecture 7

Wednesday, April 20, 2005

Supplementary Reading: Osher and Fedkiw, §14.3.4, §14.4, §14.5

In the last lecture we introduced the ENO-Roe discretization for evaluating the numerical flux function. In this lecture we introduce the ENO-Roe Fix and ENO-Local Lax Friedrichs methods for overcoming a potential problem with the ENO-Roe discretization.

1 ENO-LLF Discretization (and the Entropy Fix)

The ENO-Roe discretization can admit entropy violating expansion shocks near sonic points. That is, at a place where a characteristic velocity changes sign (a sonic point) it is possible to have a stationary expansion shock solution with a discontinuous jump in value. If this jump were smoothed out even slightly, it would break up into an expansion fan (i.e. rarefaction) and dissipate, which is the desired physical solution. For a specific cell wall, $x_{i_0+1/2}$, if there are no nearby sonic points, then we use the ENO-Roe discretization. Otherwise, we add high order dissipation to our calculation of $\mathcal{F}_{i_0+1/2}$ to break up any entropy violating expansion shocks. We call this entropy fixed version of the ENO-Roe discretization ENO-Roe Fix or just ENO-RF. More specifically, we use $\lambda_{i_0} = f'(u_{i_0})$ and $\lambda_{i_0+1} = f'(u_{i_0+1})$ to decide if there are sonic points in the vicinity. If λ_{i_0} and λ_{i_0+1} agree in sign, we use the ENO-Roe discretization where $\lambda_{i_0+1/2}$ is taken to be the same sign as λ_{i_0} and λ_{i_0+1}. Otherwise we use the ENO-LLF entropy fix discretization given below. Note that ENO-LLF is applied at both expansions where $\lambda_{i_0} < 0$ and $\lambda_{i_0+1} > 0$ and at shocks where $\lambda_{i_0} > 0$ and $\lambda_{i_0+1} < 0$. While this adds extra numerical dissipation at shocks, it is not harmful as shocks are self-sharpening. In fact, this extra dissipation provides some viscous regularization which is especially desireable in multiple spatial dimensions. For this reason, authors sometimes use the ENO-LLF method everywhere as opposed to mixing in ENO-Roe discretizations where the upwind direction is well determined by the eigenvalues λ.

The ENO-LLF discretization is formulated as follows. Consider two primitive functions H^+ and H^-. We compute a divided difference table for each of them with their first divided differences being

$$D_i^1 H^{\pm} = f(u_i) \pm \alpha_{i_0+\frac{1}{2}} u_i \tag{1}$$

where

$$\alpha_{i_0+\frac{1}{2}} = \max\left(|\lambda_{i_0}|, |\lambda_{i_0+1}|\right) \tag{2}$$

is our dissipation coefficient, and controls the amount of dissipation added. Note that the dissipation coefficient, $\alpha_{i_0+\frac{1}{2}}$, is determined *locally* for each cell wall, hence the name ENO-*local* Lax Friedrichs. (One could also construct a scheme where a global dissipation coefficient α is used, but this generally adds too much dissipation).

The second and third divided differences, $D^2_{i+1/2}H^{\pm}$ and $D^3_i H^{\pm}$ are then defined in the standard way, like those of H.

For H^+, set $k = i_0$. Then, replacing H with H^+ everywhere, define $Q_1(x)$, $Q_2(x)$, $Q_3(x)$, and finally $\mathcal{F}^+_{i_0+1/2}$ using the ENO-Roe algorithm above. For H^-, set $k = i_0 + 1$. Then, replacing H with H^- everywhere, define $Q_1(x)$, $Q_2(x)$, $Q_3(x)$, and finally $\mathcal{F}^-_{i_0+1/2}$ again by using the ENO-Roe algorithm above. Finally,

$$\mathcal{F}_{i_0+1/2} = \frac{\mathcal{F}^+_{i_0+1/2} + \mathcal{F}^-_{i_0+1/2}}{2} \tag{3}$$

is the new numerical flux function with added high order dissipation.

2 Multiple Spatial Dimensions

In multiple spatial dimensions, the ENO discretization is applied independently using a dimension by dimension discretization. For example, consider a two dimensional conservation law

$$u_t + f(u)_x + g(u)_y = 0 \tag{4}$$

on a rectangular 2-D grid. Here, we sweep through the grid from bottom to top performing ENO on 1-D horizontal rows of grid points to evaluate the $f(u)_x$ term. The $g(u)_y$ term is evaluated in a similar manner sweeping through the grid from left to right performing ENO on 1-D vertical rows of grid points. Once we have a numerical approximation to each of the spatial terms, we update the entire equation in time with a method of lines approach using, for example, a TVD Runge-Kutta method.

We emphasize that "dimension by dimension" discretization is *not* the same as "dimensional splitting", such as the first order Godunov splitting and second order Strang splitting. In dimension by dimension discretization, the fluxes in each dimension are evaluated independently, but the time stepping is still coupled.

3 Systems of Conservation Laws

In general, a hyperbolic system will simultaneously contain a mixture of processes: smooth bulk convection and wave motion, and discontinuous processes involving

contacts, shocks and rarefactions. For example, if a gas in a tube is initially prepared with a jump in the states (density, velocity and temperature) across some surface, as the evolution proceeds in time these jumps will break up into a combination of shocks, rarefactions and contacts, in addition to any bulk motion and sound waves that may exist or develop.

The hyperbolic systems we encounter in physical problems are written in what are effectively the mixed variables where the apparent behavior is quite complicated. A transformation is required to decouple them back into unmixed fields that exhibit the pure contact, shock and rarefaction phenomena (as well as bulk convection and waves). In a real system, this perfect decoupling is not possible because the mixing is nonlinear, but it can be achieved approximately over a small space and time region, and this provides the basis for the theoretical understanding of the structure of general hyperbolic systems of conservation laws. This is called a transformation to characteristic variables. As we shall see, this transformation also provides the basis for designing appropriate numerical methods.

Consider a simple hyperbolic system of N equations

$$\vec{U}_t + [\vec{F}(\vec{U})]_x = 0 \tag{5}$$

in one spatial dimension. The basic idea of characteristic numerical schemes is to transform this nonlinear system to a system of N (nearly) independent scalar equations of the form

$$u_t + \lambda u_x = 0 \tag{6}$$

and discretize each scalar equation independently in an upwind biased fashion using the characteristic velocity λ. Then transform the discretized system back into the original variables.