Lecture 8

Monday, April 25, 2005

Supplementary Reading: Osher and Fedkiw, §14.5.1

In the last lecture we started discussing systems of conservation laws. In particular consider a hyperbolic system of conservation laws with N equations in one spatial dimension, given by

$$\vec{U}_t + [\vec{F}(\vec{U})]_x = 0. \tag{1}$$

The idea is to decompose the system into N separate scalar equations of the form

$$u_t + \lambda u_x = 0.$$

1 Example

We start with an example of two separate scalar equations and show how we can change variables to write them as a coupled system. Consider the two equations

$$\begin{cases} u_t - u_x = 0 \\ v_t + v_x = 0 \\ u(x,0) = u_0(x) \\ v(x,0) = v_0(x) \end{cases}$$

The solution is

$$u(x,t) = u_0(x+t)$$
$$v(x,t) = v_0(x-t)$$

For example, figure 1 depicts the solution for the initial data given below.

$$u_0(x) = \begin{cases} 1, & x \in (-1,0) \\ 0, & \text{otherwise} \end{cases}$$
$$v_0(x) = \begin{cases} 1, & x \in (0,1) \\ 0, & \text{otherwise} \end{cases}$$

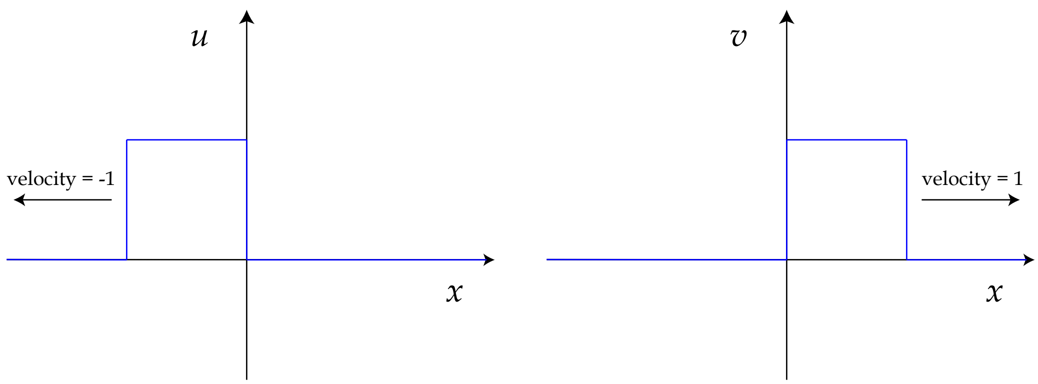

Figure 1: The solution is the initial data for u moving to the left with speed 1, and the initial data for v moving to the right with speed 1.

Next we make the change of variables

$$w = v + u$$
$$z = v - u$$

This gives

$$w_t = v_t + u_t = -v_x + u_x = -z_x$$
$$z_t = v_t - u_t = -v_x - u_x = -w_x$$

So u and v are independent of each other, but w and z depend on each other. The system for w and z can be written as

$$\begin{pmatrix} w \\ z \end{pmatrix}_t + \begin{pmatrix} z \\ w \end{pmatrix}_x = 0.$$

The solution is given by

$$
\begin{aligned}
w(x,t) &= v_0(x-t) + u_0(x+t) \\
z(x,t) &= v_0(x-t) - u_0(x+t)
\end{aligned}
$$

The graph for w is shown in figure 2.

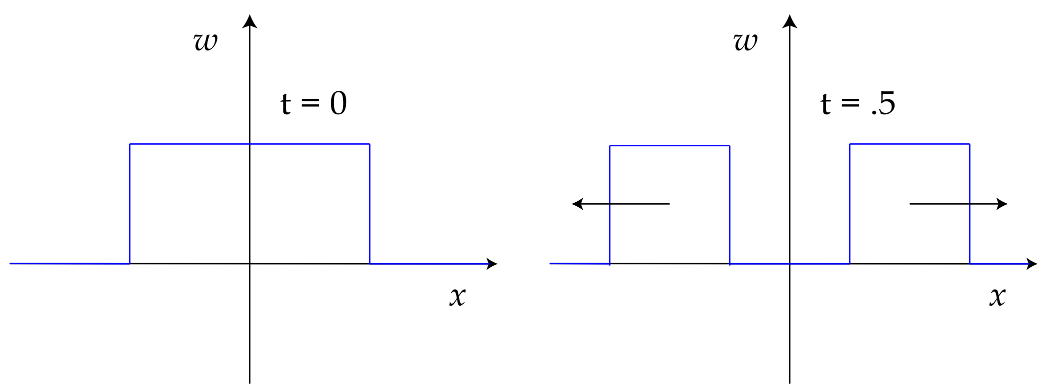

Figure 2: The solution consists of two separate components, one moving to the left, and the other moving to the right.

This demonstrates though the picture for w may appear complicated, the underlying solutions u and v are simply two waves moving to the left and right.

Now we rewrite the system as

$$
\begin{pmatrix} w \\ z \end{pmatrix}_t + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix}_x = 0
$$

which is in the form

$$
\vec{U}_t + J\vec{U}_x = 0.
$$

Similarly, we can write the system (1) in quasilinear form as

$$\vec{U}_t + \vec{F}'\left(\vec{U}\right)\vec{U}_x = 0.$$

Here $J = \frac{\partial \vec{F}}{\partial \vec{U}}$. Recall that in the scalar case

$$u_t + f\left(u\right)_x = 0$$

where we had the quasilinear form

$$u_t + f'\left(u\right)u_x = 0$$

the characteristic speed was given by $f'(u)$. For the case of systems, the characteristic speeds are given by the eigenvalues of the Jacobian, J.

Coming back to our example, we have

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

We compute the eigenvalues:

$$\det\left(\lambda I - J\right) = \begin{vmatrix} \lambda & -1 \\ -1 & \lambda \end{vmatrix} = \lambda^2 - 1$$

So the eigenvalues of J are

$$\lambda^1 = -1, \lambda^2 = 1.$$

Next we determine the eigenvectors. For $\lambda^1 = -1$, we have

$$JR^1 = \lambda^1 R^1$$

We solve for $R^1 = \begin{pmatrix} a \\ b \end{pmatrix}$.

$$\begin{aligned} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} &= -\begin{pmatrix} a \\ b \end{pmatrix} \\ \Rightarrow \begin{pmatrix} b \\ a \end{pmatrix} &= -\begin{pmatrix} a \\ b \end{pmatrix} \end{aligned}$$

Hence $R^1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is a solution.

For $\lambda^2 = 1$, we have

$$JR^2 = \lambda^2 R^2$$

We solve for $R^2 = \begin{pmatrix} c \\ d \end{pmatrix}$.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} d \\ c \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$$

Hence $R^2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a solution. Therefore, we have computed that

$$J\left(R^1, R^2\right) = \left(R^1, R^2\right) \begin{pmatrix} \lambda^1 & 0 \\ 0 & \lambda^2 \end{pmatrix}$$

or,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Note that if R is a matrix whose columns are the right eigenvectors of J,

$$JR = R\Lambda$$

then,

$$R^{-1}J = \Lambda R^{-1}$$

So the rows of R^{-1} are left eigenvectors of J. We use L to denote the matrix whose rows are the left eigenvalues of J, and we can choose it to be equal to R^{-1}, so that $LR = RL = I$.

In summary, we have computed the eigensystem for our example, and we can use this to transform J into diagonal form,

$$LJR = \Lambda.$$

It is important to note that if a system is hyperbolic, J will have N real eigenvalues λ^p, $p = 1, \ldots, N$, and N linearly independent right eigenvectors. Then, once the eigensystem is determined, we can use it to diagonalize the matrix J.

Suppose we want to discretize our equation at the node x_0, where L and R have values L_0 and R_0. To get a locally diagonalized form, we multiply our system equation by the constant matrix L_0 which nearly diagonalizes J over the region near x_0. We require a constant matrix so that we can move it inside all derivatives to obtain

$$[L_0\vec{U}]_t + L_0JR_0[L_0\vec{U}]_x = 0 \tag{2}$$

where we have inserted $I = R_0L_0$ to put the equation in a more recognizable form. The spatially varying matrix L_0JR_0 is exactly diagonalized at the point

x_0, with eigenvalues λ_0^p, and it is nearly diagonalized at nearby points. Thus the equations are sufficiently decoupled for us to apply upwind biased discretizations independently to each component with λ_0^p determining the upwind biased direction for the p-th component equation. Once this system is fully discretized, we multiply the entire system by $L_0^{-1} = R_0$ to return to the original variables.

In terms of our original equation 1, our procedure for discretizing at a point x_0 is simply to multiply the entire system by the left eigenvector matrix L_0,

$$[L_0\vec{U}]_t + [L_0\vec{F}(\vec{U})]_x = 0 \tag{3}$$

and discretize the $p = 1, \ldots, N$ scalar components of this system

$$[(L_0\vec{U})_p]_t + [(L_0\vec{F}(\vec{U}))_p]_x = 0 \tag{4}$$

independently, using upwind biased differencing with the upwind direction for the p-th equation determined by the sign of λ^p. We then multiply the resulting spatially discretized system of equations by R_0 to recover the spatially discretized fluxes for the original variables

$$\vec{U}_t + R_0\Delta(L_0\vec{F}(\vec{U})) = 0 \tag{5}$$

where Δ stands for the upwind biased discretization operator, i.e. either the ENO-RF or ENO-LLF discretization.

We call λ^p the p-th characteristic velocity or speed, $(L_0\vec{U})_p = \vec{L}_0^p \cdot \vec{U}$ the p-th characteristic state or field (here L^p denotes the p-th row of L, i.e. the p-th left eigenvector of J), and $(L_0\vec{F}(\vec{U}))_p = \vec{L}_0^p \cdot \vec{F}(\vec{U})$ the p-th characteristic flux. According to the local linearization, it is approximately true that the p-th characteristic field rigidly translates in space at the p-th characteristic velocity. Thus this decomposition corresponds to the local physical propagation of independent waves or signals.