Lecture 9

Wednesday, April 27, 2005

Supplementary Reading: Osher and Fedkiw, §14.5.2

1 Discretization

We are interested in constructing a discretization for the hyperbolic system of conservation laws

$$\vec{U}_t + \vec{F}\left(\vec{U}\right)_x = 0. \tag{1}$$

Assume that the system (1) has N equations. Then the Jacobian, $J = \frac{\partial \vec{F}}{\partial \vec{U}}$, will be and $N \times N$ matrix. Futhermore, we know that J is diagonalizable, since the system is hyperbolic. Let us denote the eigenvalues, right eigenvectors, and left eigenvectors of J as λ^p, R^p, L^p respectively for $p = 1, \ldots, N$. Recall from the previous lecture that we can choose the left and right eigenvectors so that for

$$R = \left(R^1\ R^2\ \ldots\ R^N\right), \qquad L = \begin{pmatrix} L^1 \\ L^2 \\ \vdots \\ L^N \end{pmatrix}$$

the relation $RL = LR = I$ holds. That is, R and L are chosen to be inverses.

In the previous lecture we looked at a linear, constant coefficient system. In that case, the Jacobian was a constant matrix. In general, the Jacobian, and hence its eigensystem, will be spatially varying.

As in the scalar case, our discretization is of the form

$$\left(\vec{U}_i\right)_t + \frac{\vec{F}_{i+1/2} - \vec{F}_{i-1/2}}{\triangle t} = 0.$$

For grid point i_0, we need to compute the numerical flux functions at $x_{i_0+1/2}$ and $x_{i_0-1/2}$. Let us look in detail at computing $F_{i_0+1/2}$.

The first step is to evaluate the eigensystem at the point $x_{i_0+1/2}$. Since we only have $\vec{U}$ at the grid points, we obtain $\vec{U}$ at the cell walls using the standard average, $\vec{U}_{i_0+1/2} = (\vec{U}_{i_0} + \vec{U}_{i_0+1})/2$. Then for p = 1, ..., N, we find the component of the numerical flux function in the p-th characteristic field. In the p-th characteristic field we have an eigenvalue $\lambda^p(\vec{U}_{i_0+1/2})$, left

eigenvector $\vec{L}^p(\vec{U}_{i_0+1/2})$, and right eigenvector $\vec{R}^p(\vec{U}_{i_0+1/2})$.We put $\vec{U}$ values and $\vec{F}(\vec{U})$ values into the p-th characteristic field by taking the dot product with the left eigenvector,

$$u_{i_0} = \vec{L}^p(\vec{U}_{i_0+1/2}) \cdot \vec{U}_{i_0} \tag{2}$$

$$f_{i_0} = \vec{L}^p(\vec{U}_{i_0+1/2}) \cdot \vec{F}(\vec{U_{i_0}}) \tag{3}$$

where u_{i_0} and f_{i_0} are scalars. Once in the characteristic field we perform a scalar version of the conservative ENO scheme obtaining a scalar numerical flux function $\mathcal{F}_{i_0+1/2}$ in the scalar field. We take this flux out of the characteristic field by multiplying with the right eigenvector,

$$\vec{\mathcal{F}}^p_{i_0+1/2} = \mathcal{F}_{i_0+1/2}\vec{R}^p(\vec{U}_{i_0+1/2}) \tag{4}$$

where $\vec{\mathcal{F}}^p_{i_0+1/2}$ is the portion of the numerical flux function $\vec{\mathcal{F}}_{i_0+1/2}$ from the p-th field. Once we have evaluated the contribution to the numerical flux function from each field, we get the total numerical flux by summing the contributions from each field,

$$\vec{\mathcal{F}}_{i_0+1/2} = \sum_p \vec{\mathcal{F}}^p_{i_0+1/2} \tag{5}$$

completing the evaluation of our numerical flux function at the point $x_{i_0+1/2}$.

2 Shallow Water Equations

The shallow water equations are given by

$$\begin{pmatrix} h \\ hu \end{pmatrix}_t + \begin{pmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \end{pmatrix}_x = 0. \tag{6}$$

where h is the height of the water, and u is the velocity. The first equation is the equation for conservation of mass, and the second is the equation for conservation of momentum. In order to discretize the system using the procedure we described above, we must first find the Jacobian and its eigensystem analytically.

In computing the Jacobian, it is very important to remember that we take the conserved variables (in this case h and hu) to be the independent variables. To make this fact more apparent, we can rewrite the equations as

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_t + \begin{pmatrix} u_2 \\ u_2^2 u_1^{-1} + \frac{1}{2}gu_1^2 \end{pmatrix}_x = 0. \tag{7}$$

and then define $h = u_1$, $u = u_2 u_1^{-1}$. Below we compute the Jacobian matrix.

$$\begin{aligned}
J &= \frac{\partial \vec{F}}{\partial \vec{U}} \\
&= \begin{pmatrix} \frac{\partial \vec{F}_1}{\partial u_1} & \frac{\partial \vec{F}_1}{\partial u_2} \\ \frac{\partial \vec{F}_2}{\partial u_1} & \frac{\partial \vec{F}_2}{\partial u_2} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\partial (hu)}{\partial h} & \frac{\partial (hu)}{\partial (hu)} \\ \frac{\partial}{\partial h}\left((hu)^2 h^{-1} + \frac{1}{2} g h^2 \right) & \frac{\partial}{\partial (hu)}\left((hu)^2 h^{-1} + \frac{1}{2} g h^2 \right) \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ -(hu)^2 h^{-2} + gh & 2(hu) h^{-1} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ -u^2 + gh & 2u \end{pmatrix}
\end{aligned}$$

Note: if you find the treatment of h and hu as independent variables in the above computation confusing, you may prefer rewrite the system as in (7), compute the Jacobian in terms of u_1 and u_2, and then substitute for h and u at the end.

Next we find the eigensystem for the Jacobian. We have

$$\begin{aligned}
\det(\lambda I - J) &= \begin{vmatrix} \lambda & -1 \\ -u^2 + gh & \lambda - 2u \end{vmatrix} \\
&= \lambda^2 - 2u\lambda + u^2 - gh
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \lambda &= \frac{2u \pm \sqrt{4u^2 - 4u^2 + 4gh}}{2} \\
&= u \pm \sqrt{gh}
\end{aligned}$$

Next we find the right eigenvectors.

$$\begin{aligned}
J \begin{pmatrix} a \\ b \end{pmatrix} &= \left(u \pm \sqrt{gh} \right) \begin{pmatrix} a \\ b \end{pmatrix} \\
\Rightarrow \begin{pmatrix} b \\ \left(-u^2 + gh \right) a + 2ub \end{pmatrix} &= \left(u \pm \sqrt{gh} \right) \begin{pmatrix} a \\ b \end{pmatrix}
\end{aligned}$$

Therefore, we have

$$R^1 = \begin{pmatrix} 1 \\ u + \sqrt{gh} \end{pmatrix}, \qquad R^2 = \begin{pmatrix} 1 \\ u - \sqrt{gh} \end{pmatrix}$$

Then

$$R = \begin{pmatrix} R^1 & R^2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ u + \sqrt{gh} & u - \sqrt{gh} \end{pmatrix}$$

Hence

$$L = R^{-1} = \frac{1}{-2\sqrt{gh}} \begin{pmatrix} u - \sqrt{gh} & -1 \\ -u - \sqrt{gh} & 1 \end{pmatrix}$$

or

$$L^1 = \left(-\frac{u}{2\sqrt{gh}} + \frac{1}{2}, \quad \frac{1}{2\sqrt{gh}} \right), \qquad L^2 = \left(\frac{u}{2\sqrt{gh}} + \frac{1}{2}, \quad -\frac{1}{2\sqrt{gh}} \right).$$

3 Compressible Flow

The inviscid Euler equations for one phase compressible flow in the absence of chemical reactions in one spatial dimension are

$$\vec{U}_t + \vec{F}(\vec{U})_x = 0 \tag{8}$$

which can be written in detail as

$$\begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 + p \\ (E + p)u \end{pmatrix}_x \tag{9}$$

where ρ is the density, u are the velocities, E is the total energy per unit volume, and p is the pressure. The total energy is the sum of the internal energy and the kinetic energy,

$$E = \rho e + \rho(u^2)/2 \tag{10}$$

where e is the internal energy per unit mass.

3.1 Ideal Gas Equation of State

For an ideal gas

$$p = \rho R T$$

where $R = R_u/M$ is the specific gas constant with $R_u \approx 8.31451 J/(molK)$ the universal gas constant and M the molecular weight of the gas. Also valid for an ideal gas is

$$c_p - c_v = R$$

where c_p is the specific heat at constant pressure and c_v is the specific heat at constant volume. Gamma is the ratio of specific heats,

$$\gamma = c_p/c_v.$$

For an ideal gas, one can write

$$de = c_v dT \tag{11}$$

and assuming that c_v does not depend on temperature (calorically perfect gas), integration yields

$$e = e_o + c_v T \tag{12}$$

where e_o is not uniquely determined, and one could choose any value for e at $0K$. We take $e_0 = 0$ arbitrarily for simplicity.

Note that

$$p = \rho RT = \frac{R}{c_v}\rho e = \frac{c_p - c_v}{c_v}\rho e = \rho(\gamma - 1)e = (\gamma - 1)\rho e$$

So, our equation of state is

$$p = (\gamma - 1)\rho e,$$

or

$$p = (\gamma - 1)\left(E - \frac{\rho u^2}{2}\right).$$