Homework 5

I. Linear Hyperbolic Systems

Consider the Linearized Gas Dynamics equations, given by

$$\begin{pmatrix} \rho \\ u \end{pmatrix}_t + \begin{pmatrix} 0 & b \\ \frac{a^2}{b} & 0 \end{pmatrix} \begin{pmatrix} \rho \\ u \end{pmatrix}_x = 0.$$

A.

Find the eigenvalues and right and left eigenvectors of the Jacobian for the system.

B.

Find the solution of the above system given the initial conditions

$$\begin{cases} \rho(x,0) = \rho_0(x) \\ u(x,0) = u_0(x) \end{cases}$$

II. ENO-LLF for Systems

The shallow water equations are

$$\begin{pmatrix} h \\ hu \end{pmatrix}_t + \begin{pmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \end{pmatrix}_x = \mathbf{0}.$$

Recall from class that the associated Jacobian eigenvalues are

$$\lambda_1 = u + \sqrt{gh}, \lambda_2 = u - \sqrt{gh}$$

with left eigenvectors

$$\mathbf{L}_1 = \left(\frac{\sqrt{gh}-u}{2\sqrt{gh}}, \frac{1}{2\sqrt{gh}} \right), \mathbf{L}_2 = \left(\frac{\sqrt{gh}+u}{2\sqrt{gh}}, -\frac{1}{2\sqrt{gh}} \right)$$

and right eigenvectors

$$\mathbf{R}^1 = \begin{pmatrix} 1 \\ u + \sqrt{gh} \end{pmatrix}, \mathbf{R}^2 = \begin{pmatrix} 1 \\ u - \sqrt{gh} \end{pmatrix}$$

Now, assume we want to update point h and u at x_j. Fill in the missing steps in the following algorithm. We first compute $\mathbf{F}_{j+\frac{1}{2}}$. We define $s1$ and $f1$ as

$$s1 = ???, f1 = ???$$

and similarly $s2$ and $f2$ as

$$s2 = ???, f2 = ???.$$

Then, to perform ENO-LLF we need to evaluate

$$D1^1_i H^+ = \frac{1}{2} f1_i + \frac{1}{2}\alpha 1_{j+\frac{1}{2}} s1_i$$

$$D1^1_i H^- = \frac{1}{2} f1_i - \frac{1}{2}\alpha 1_{j+\frac{1}{2}} s1_i$$

where

$$\alpha 1_{j+\frac{1}{2}} = ???.$$

And similarly,

$$D2^1_i H^+ = \frac{1}{2} f2_i + \frac{1}{2}\alpha 2_{j+\frac{1}{2}} s2_i$$

$$D2_i^1 H^- = \frac{1}{2} f2_i - \frac{1}{2} \alpha 2_{j+\frac{1}{2}} s2_i$$

where

$$\alpha 2_{j+\frac{1}{2}} = ???.$$

We then set

$$F1^+_{j+\frac{1}{2}} = D1^1_j H^+, F1^-_{j+\frac{1}{2}} = D1^1_{j+1} H^-$$

and

$$F2^+_{j+\frac{1}{2}} = D2^1_j H^+, F2^-_{j+\frac{1}{2}} = D2^1_{j+1} H^-$$

and finally

$$F1_{j+\frac{1}{2}} = F1^+_{j+\frac{1}{2}} + F1^-_{j+\frac{1}{2}}$$

$$F2_{j+\frac{1}{2}} = F2^+_{j+\frac{1}{2}} + F2^-_{j+\frac{1}{2}}.$$

These fluxes are then used as

$$\mathbf{F}_{j+\frac{1}{2}} = ???.$$

$\mathbf{F}_{j-\frac{1}{2}}$ is computed similarly (e.g. just plug in $j-1$ in place of j in the previous calculation). $\mathbf{F}_{j+\frac{1}{2}}$ and $\mathbf{F}_{j-\frac{1}{2}}$ can then be used to update u_j^n and h_j^n as

$$\begin{pmatrix} u_j^{n+1} \\ h_j^{n+1} u_j^{n+1} \end{pmatrix} = \begin{pmatrix} u_j^n \\ h_j^n u_j^n \end{pmatrix} - \frac{\Delta t}{\Delta x} (\mathbf{F}_{j+\frac{1}{2}} - \mathbf{F}_{j-\frac{1}{2}}).$$