Homework 6

I.

Compute the Jacobian for the $1D$ Euler equations presented in class. You may assume the ideal gas equation of state given by $p(\rho, e) = (\gamma - 1)\rho e$.

II.

We show in class that for 1D incompressible flow the inviscid Euler equations decouple to:

$$\begin{aligned} \rho_t + u\rho_x &= 0 \\ u_t + \frac{p_x}{\rho} &= 0 \\ e_t + ue_x &= 0 \end{aligned}$$

The $3D$ Euler equations are given by

$$\begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ E \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho uw \\ (E+p)u \end{pmatrix}_x + \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ \rho vw \\ (E+p)v \end{pmatrix}_y + \begin{pmatrix} \rho w \\ \rho uw \\ \rho vw \\ \rho w^2 + p \\ (E+p)w \end{pmatrix}_z = 0$$

where ρ is the density, $\mathbf{v} = (u, v, w)$ are the velocities, E is the total energy per unit volume, and p is the pressure. The total energy is the sum of the internal energy and the kinetic energy,

$$\begin{aligned} E &= \rho\left(e + \frac{1}{2}\|\mathbf{v}\|^2\right) \\ &= \rho e + \rho(u^2 + v^2 + w^2)/2 \end{aligned}$$

where e is the internal energy per unit mass. The assumption of incompressibility gives

$$\nabla \cdot \mathbf{v} = u_x + v_y + w_z = 0.$$

Show that in 3D the inviscid Euler equations with the assumption of incompressible flow decouple to:

$$\begin{aligned} \rho_t + \mathbf{v} \cdot \nabla\rho &= 0 \\ u_t + \mathbf{v} \cdot \nabla u + \frac{p_x}{\rho} &= 0 \\ v_t + \mathbf{v} \cdot \nabla v + \frac{p_y}{\rho} &= 0 \\ w_t + \mathbf{v} \cdot \nabla w + \frac{p_z}{\rho} &= 0 \\ e_t + \mathbf{v} \cdot \nabla e &= 0 \end{aligned}$$

where $\mathbf{v} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$.